CORRECTION

"In the Bibliography to the paper by M. Morse and W. Transue, A Calculus of Frechet Variations, Vol. XIV, Nos. 2 and 3, 1950:

Papers by Morse and Transue should be lettered MT (1) to MT (10).

The paper by G. H. Hardy appears in the *Messenger of Mathematics*, 49 (1919-1920), 149-155."

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S. S. Pillai
1901—1950

THIS VOLUME

IS DEDICATED TO

S. S. Pillai

IN MEMORY OF HIS ACHIEVEMENTS

IN MATHEMATICAL RESEARCH

S. S. PILLAI

By K. CHANDRASEKHARAN

The death of Dr. S. S. Pillai (Subbayya Pillai Sivasankaranarayana Pillai) in the crash of the aircraft, Star of Maryland, near Cairo on August 31, 1950 removes from our midst one of our best mathematicians. Pillai was on his way to the United States when the accident occurred. He had planned to spend a year at the Institute for Advanced Study, Princeton, and was due to attend the International Congress of Mathematicians at Harvard University from August 30 to September 6, 1950, as a delegate of the University of Madras. The news of the tragedy reached the Congress on the second day of its session, and mathematicians of many lands were deeply affected by it. It is probably the first instance of the death of an outstanding mathematician in an accident of civil aviation.

Sivasankaranarayana Pillai was born on April 5, 1901 at Vallam, near Courtalam, in the Tirunelveli district of the Presidency of Madras*. His father, Subbayya Pillai, was a 'Government Contractor'. His mother, Gomathi Ammal, passed away within a year of his birth, and an old relative of the family served as his foster-mother. When he grew to be five, his father arranged for his tuition at home, and he received instruction that way for about three years. He went to the Middle School at Shencottah when he was about nine, and this involved considerable difficulty. But he made good progress, and his talents earned for him the lasting affection of one of his teachers whom he called Sastriar. In due course, he moved on to the local High School; when he was in the Matriculation class, his father passed away. He needed money to continue his education, and for a boy under eighteen, even borrowing was very difficult. Sastriar came to his aid, and helped him finish High School. Pillai then got a scholarship which, combined with Sastriar's help, enabled him to go to the Scott Christian College at Nagercoil, where he did his Intermediate. He did his B.A. at Maharajah's College, Trivandrum, and moved to Madras around 1927. He received a stipendiary Research Studentship from the University of Madras, and it was here that his first contacts with higher mathematics were established, with the guidance of Professor K. Ananda-Rau,

* I am indebted to Mrs. S. S. Pillai for biographical information regarding the late Dr. Pillai.

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who officially supervised his work, and of Dr. R. Vaidyanathaswamy. Pillai's early papers bear witness to the influence exercised on him by both these mathematicians, though the nature of his pursuits rendered Ananda-Rau's influence inevitably greater. One recalls Pillai's frequent references in later years to Ananda-Rau's "inspiring guidance". During his stay at Madras he wrote many papers, and three of them constituted his Thesis for the M.Sc. degree. In 1929 he was appointed a Lecturer at Annamalai University, and during the same year he married Srimathi Sundaram. It was during his stay at Annamalai University that he did his best work, and got the D.Sc. degree from the University of Madras. He was the first to receive a doctor's degree from that University for work in mathematics. In 1941, he went to Travancore with high hopes of building up a school of mathematics at the newly created University, and if he had stayed there long enough he would have probably succeeded. In 1942, however, he was appointed a Lecturer at Calcutta University, because of the intensity of interest evinced in his behalf by Professor F. W. Levi. Unluckily for Pillai, it was only a little later that Calcutta became a centre of strategic impor tance in the war against Japan on the one hand, and of political upheaval and communal turmoil on the other. At Calcutta, perhaps for the first time, he developed an interest in the theory of functions of a real variable and in the theory of the foundations of mathematics. These were very much on his mind during his last days. He had a brilliant pupil at Calcutta by name L. G. Sathe who did excellent work under his direction (the best part of which has been submitted for publication in the Transactions of the American Mathematical Society), but unluckily again, Sathe's mathematical researches have since been cut off by a cruel illness. During the last few years, Pillai was trying to solve the famous unsolved problem in the theory of Fourier Series: Does the Fourier Series of a continuous periodic function necessarily have a point of convergence? He was also interesting himself in a new type of transfinite cardinal arithmetic. Obviously he needed a change, or rather a fresh stimulus, and this came to him in the form of a stipend from Princeton. Friends of his had great expectations that his contacts with mathematicians in the United States would bring forth fresh contributions from him. But tragedy has destroyed all such expectations. Pillai has left behind his wife and three sons.

Pillai's abiding interest was the theory of numbers, and he published about seventy papers on the subject, varying in size, in importance and in interest. The volume of his work is therefore quite large; what is even more remarkable about it is that "it

is fresh and has a distinct touch of originality". Pillai was a lover of problems, and of genuinely difficult problems. The theory of numbers has always had an abundant supply of problems, and so he was never forced to expand his interests. This resulted in a concentration of effort that is well typified in his mastery of what one may call asymptotic analysis. He was a great admirer of Hardy and Littlewood, of Landau and Vinogradov. His work reveals his remarkable capacity for handling Orders of Infinity, Approximations and Inequalities. Every paper of his has at least one piece of original argument, and his best papers borrow very little from others. As he grew older, he perhaps developed an eagerness to publish his papers faster ; but this eagerness was not an innate quality, it grew out of his experience in connexion with the solution of Waring's problem, which revealed to him how close the finish could be sometimes. To prove this point, one has only to notice that the paper on highly composite numbers which he had written in 1933 was published by him only in 1944, when Erdos and Alaoglu had discovered similar results.

While it is not possible here to review his work in great detail, it is perhaps fitting that something ought to be said about his contribution to the solution of Waring's problem, which is almost certainly his best piece of work, and one of the very best achievements in Indian Mathematics since Ramanujan. One can rely on Pillai's own description. [*Cf.* his paper (43)].

'In the year 1640, Fermat stated, among other results, that every number is the sum of four squares. Though his statement implied that he had a proof by his famous method of infinite descent, it was Lagrange who published a proof for it in 1770. Suppose we want to express n as a sum of squares, say, in the form $n = x_1^2 + \dots + x_s^2$. To start with, we have no reason to expect that the minimum value of s will not increase with n. Fermat's theorem means that it is possible to find s independent of n, and its value is 4. This is certainly one of the most beautiful results in Arithmetic.

'Waring's problem is the generalization of this result for k-th powers. In the year 1770, Waring, Lucasian Professor of Mathematics in Cambridge University, stated, in his *Meditations Algebraic*, that every number is the sum of 4 squares, 9 cubes, 19 biquadrates and so on. Let $g(k)$ be the least value of s such that every number is the sum of s, k-th powers ; that is, the equation $n = x_1^k + \dots + x_s^k$ is solvable for every n. Here also, in the first instance, we do not see whether the least value of s will

increase with n or not. Waring's assertion means not only that $g(k)$ is independent of n, but also that $g(2) = 4$, $g(3) = 9$, $g(4) = 19$, and so on. But Waring's result was only a conjecture, and he had no proof of it.

'In 1772, Euler, the son of the famous Leonhard Euler, showed that $l.2^k-1$ requires 2^k+l-2, kth powers, where $l=[(3/2)]^k$ and thereby proved that $g(k) \geqslant 2^k+l-2$. The next advance was made in 1858 when Liouville proved that $g(4) \leqslant 53$. In the years up to 1909 upper bounds for $g(k)$ were evaluated and improved for all k up to 10 excepting 9.

'The year 1909 is an important one in the history of the problem. In that year, Wiefrich proved that $g(3) = 9$. Landau proved that every large number is the sum of eight cubes and above all Hilbert proved the existence of $g(k)$. Speaking about the problem and its solution by Hilbert, Professor Hardy says "within the limits which it has set for itself, it is absolutely and triumphantly successful, and it stands with the work of Hadamard and de la Vallee Poussin in the theory of primes as one of the landmarks in the modern history of the theory of numbers".

'Landau's result is important in that it was responsible for the introduction of another and more fundamental function $G(k)$ by Hardy and Littlewood. Landau's 'singularly beautiful theorem' means that the number of numbers which require nine cubes is finite. As a matter of fact, recently Dickson has proved that 23 and 239 are the only numbers which require as many as 9 cubes. $G(k)$ is defined as the least value of s such that every sufficiently large number (every number from a certain integer onwards) is the sum of s, kth powers. Since numbers of the form $8m+7$ cannot be expressed as the sum of three squares, Fermat's result means that $g(2) = G(2) = 4$. Landau's result takes the form $G(3) \leqslant 8$.

'Hilbert's proof was a pure existence proof. The next advance was due to Hardy and Littlewood. In the years from 1919 to 1927 they attacked the problem in a series of memoirs. They discovered a method of applying Cauchy's theorem to the problem. It may be remarked that the 'Farey-dissection' of the circle, which plays a fundamental role in all subsequent investigations, first appears in Hardy-Ramanujan's memoir on Partitions. By their important method, besides finding upper bounds for $G(k)$, Hardy and Littlewood were able to find an asymptotic formula for the number of representations of a number as the sum of s, kth powers, when s is large. In these memoirs, they introduced many fundamental ideas and devices without which subsequent advances would have been impossible. The final result about $G(k)$ which they proved is that

$$\overline{\lim}\ G(k)/(k.\,2^k) \leqslant \tfrac{1}{4}.$$

'Though the success of their method was phenomenal in dealing with $G(k)$, they did not apply it to obtain an upper bound for $g(k)$. But Vinogradov found out a new method to prove Hilbert's theorem, and the consequences of that method are given in Landau's *Vorlesungen uber Zahlentheorie*. The result proved there is that

$$\overline{\lim}\ g(k)/(k.\,2^k) \leqslant \tfrac{1}{2}.$$

'In 1933, R. D. James applied the Hardy-Littlewood method and reduced the constant in the above limit to 1/4, and improved upon the upper bounds for $g(k)$ given by earlier investigators for small values of k.

'In the year 1935, Vinogradov was able to prove the wonderful result that

$$\lim\ G(k)/(k \log k) \leqslant 6.$$

Though this is a tremendous improvement upon Hardy and Littlewood's result, it is far from the final solution of $G(k)$. Yet it prepared the ground for the solution of $g(k)$ which is final in a sense.

'By evaluating the constant in Vinogradov's paper, I was able to prove, in September 1935, that $g(k) = 2^k + (3/2)^k + O\,(\frac{4}{3})^k$. It may be mentioned that the result was proved without applying Dickson's method of ascent. The result was communicated to the conference of the Indian Mathematical Society held at Delhi in December 1935. But in the third week of December 1935, I was able to prove that if $\left\{ (3/2)^k \right\} \leqslant 1-(l+3)/2^k$ then $g(k) = 2^k+l-2$ when $k \geqslant k_0$. The result was published in the March issue of Journal of the Annamalai University, and advanced reprints were sent to some mathematicians including L. E. Dickson about the 10th of February, 1936 (perhaps before the 10th).

'By refining the above method, I brought down k_0 to 7 in the last week of January 1936, and published the proof when $k_0 \geqslant 8$ in the Journal of the Indian Mathematical Society. The fact that this problem should depend upon the fractional part of $(3/2)^k$ was a great surprise to me.

'The proof of the above result when $k_0 \geqslant 7$ together with an exact formula for $g(k)$ when $\left\{ (3/2)^k \right\} \geqslant 1-(l-1)/2^k$ was published by L. E. Dickson in the July (1936) issue of the American Journal of Mathematics. The latter result was discovered independently by me.

'These results form 'an almost complete solution' of the problem of $g(k)$ in a sense. Though there is no doubt that

$$\left\{ \left(\tfrac{3}{2}\right)^k \right\} \leqslant 1-(l+3)2^k$$

for all $k \geqslant 4$, unless it is settled one way or the other, we cannot say that the problem of $g(k)$ is completely solved.'

This 'almost complete solution' was completed by Pillai, who proved in 1940 that $g(6)=73$, and especially by I. Niven in 1944 (Amer. J. of Math., 66, 1944, 137-43).

Another piece of his work which is not as famous as his solution of the problem of $g(k)$, but which is quite well known is on Diophantine approximation and centres on the following theorem: If m, n, a, b, are given positive integers, $am^x - bn^y \neq 0$ for any integral x and y, and δ is positive, then

$$| am^x - bn^y | > m^{(1-\delta)x}$$

for all $x > x_0(\delta)$. This is very nicely described by Hardy in his *Twelve Lectures on Ramanujan.*

Pillai was a person of genuine modesty and remarkable simplicity. Not many people knew him intimately, but those who did, developed an attachment to him which bordered on affection. He was unsophisticated in some peculiar sense. On matters of public interest, however, he had strong convictions. One can recall his fierce criticism of the use of the first atomic bomb, and his dictum: quit the Commonwealth and retain the English language. He was an avowed vegetarian, who made a point of it, and one can recall the satisfaction with which he received assurances from friends that his vegetarianism could be safely kept even in the United States. He possessed that rare quality among intellectuals—intellectual honesty, which endeared him to his friends, but lost him many material advantages. Remarkable though his contribution has been to mathematics, it would have been, perhaps, even more remarkable if he had had the opportunity of staying abroad for a few years. It cannot be said that his academic preferment was justly proportiontate to his deserts. As far as one is able to ascertain, he was not a Fellow of any Academy or Institute in India. He was a member of the Indian Mathematical Society from the beginning of his mathematical career, and attended as many of its Conferences as he could. He was, for years, a valued member of the Editorial Committee of this Journal. We mourn his loss.

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A THEOREM ON THE DISTRIBUTION OF THE VALUES OF L-FUNCTIONS

By S. CHOWLA AND P. ERDOS

1. Let (d/n) [where $d \equiv 0, 1 \pmod 4$, $d \neq u^2$, u integral] be Kronecker's symbol. Define for $s > 0$

$$L_d(s) = \sum_1^\infty \left(\frac{d}{n}\right) n^{-s}.$$

Denote by $g(a, x)$ the number of positive integers $d \leqslant x$ such that

$$d \equiv 0, 1 \ (4); \ d > 0; \ d \neq u^2;$$
$$L_d(s) < a.$$

We prove the following

THEOREM. *If* $s > 3/4$ *we have*

$$\lim_{x \to \infty} \frac{g(a, x)}{x/2} = g(a) \text{ exists};$$

furthermore $g(0) = 0$, $g(\infty) = 1$ *and* $g(a)$, *the distribution function, is a continuous and strictly increasing function of* a.

It is implicit in our theorem that for almost all d [i.e. with the exception of $o(x)$ integers $d \leqslant x$] $L_d(s) > 0$ provided that $s > 3/4$. This result seems to be new.

If the extended Riemann hypothesis holds, then of course $L_d(s) > 0$ for all d.

We can also prove our theorem when d runs over negative integral values whose absolute values do not exceed x. (Similar questions on the distribution functions of number theoretic functions were considered in several papers of Wintner, e.g. *Amer. Jour. of maths.* 63, (1941), 223-248; see also Jessen-Wintner, *Trans. Amer. Math. Soc.* 38 (1935), 48-88.)

2. Write for $s > 0$,

$$L_d(s, y) = \sum_{n \leqslant y} \left(\frac{d}{n}\right) n^{-s},$$

$$L_d^{(t)}(s, y) = \sum_{\substack{n \\ n \leqslant y}} \left(\frac{d}{n}\right) n^{-s},$$

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$$L_d^{(t)}(s) = \sum_n \left(\frac{d}{n}\right) n^{-s},$$

where, in the last two summations, n runs only over positive integers whose greatest prime factor does not exceed t. Clearly

$$L_d(s, x^{2/3}) - L_d^{(t)}(s, x^{2/3}) = \sum_{\substack{n \\ n \leqslant x^{2/3},\, P(n) > t}}' \left(\frac{d}{n}\right) n^{-s},$$

where $P(n)$ denotes the greatest prime factor of n. Hence

$$| L_d(s, x^{2/3}) - L_d^{(t)}(s, x^{2/3}) |^2$$

$$= \sum_{\substack{m,\, n \leqslant x^{2/3} \\ P(m),\, P(n) > t}} \sum \left(\frac{d}{m}\right) \left(\frac{d}{n}\right) (mn)^{-s}$$

summing for all $d \equiv 0, 1 \ (4)$ which are in the range $2 \leqslant d \leqslant x$ and are not perfect squares:

$$\sum_d | L_d(s, x^{2/3}) - L_d^{(t)}(s, x^{2/3}) |^2$$

$$= \sum_d \sum_{\substack{m,\, n \leqslant x^{2/3} \\ P(m),\, P(n) > t}} \left(\frac{d}{m}\right) \left(\frac{d}{n}\right) (mn)^{-s} = \Sigma_1 + \Sigma_2,$$

where in Σ_1 the product mn is not a perfect square, while in Σ_2 the product mn is a square.

To estimate Σ_1 we use

Lemma 1. *If k is not a perfect square we have*

$$\sum_d \left(\frac{d}{k}\right) = O(k^{\frac{1}{2}} \log k) + O(x^{\frac{1}{2}}).$$

The summation is for all d with

$$2 \leqslant d \leqslant x;\ d \equiv 0, 1 \ (4);\ d \neq u^2.$$

Proof. Polya (*Gottinger Nachrichten*, 1918) proved that

$$\sum_a^b \chi(n) = O(k^{\frac{1}{2}} \log k)$$

if χ is a primitive character (mod k), $k > 1$. From this several writers deduced that this result is true for any non-principal character (mod k). We easily deduce

$$\sum_{\substack{a \\ n\equiv 0(4)}}^{b} \chi(n) = O(\sqrt{k}\log k), \sum_{\substack{a \\ n\equiv 1(4)}}^{b} \chi(n) = O(\sqrt{k}\log k)$$

which proves Lemma 1; the term $O(\sqrt{x})$ in the lemma is accounted for by the fact that the summation in our lemma excludes the d which are perfects quares.

Thus using Lemma 1 we have (since $s > \frac{3}{4}$)

$$|\Sigma_1| < c \sum_{m,n \leqslant x^{2/3}} \frac{\sqrt{(mn)}\log(mn)+\sqrt{x}}{(mn)^s}$$

$$< c\left\{x^{\frac{2}{3}(3-2s)+\varepsilon}+x^{\frac{2}{3}(2-2s)+\frac{1}{2}}\right\} = o(x). \tag{1}$$

In Σ_2, $mn = w^2$, hence

$$|\Sigma_2| \leqslant \sum_{\substack{w=1 \\ P(w)>t}}^{x^{2/3}} \frac{xd(w^2)}{w^{2s}} < \frac{cx}{\sqrt{t}}, \tag{2}$$

where $d(n)$ denotes the number of divisors of n. We used here the well-known fact that $d(n) = O(n^\varepsilon)$ for any $\varepsilon > 0$ and also that $s > \frac{3}{4}$.

Thus we obtain from (1) and (2)

Lemma 2. *Let $s > \frac{3}{4}$. Then there exists an absolute constant c so that*

$$\sum_d |L_d(s, x^{\frac{2}{3}}) - L_d^{(t)}(s, x^{\frac{2}{3}})|^2 < \frac{cx}{\sqrt{t}}.$$

3. We next estimate

$$L_d(s) - L_d(s, x^{2/3}) = \sum_{n > x^{2/3}} \left(\frac{d}{n}\right) n^{-s}$$

$$= \frac{O(\sqrt{d}\log d)}{x^{2/3s}} = O(x^{\frac{1}{2}-2/3s+\varepsilon}) = o(1). \tag{3}$$

Further we have

$$L_d^{(t)}(s) - L_d^{(t)}(s, x^{\frac{2}{3}}) = \sum_{\substack{n \geqslant x^{2/3} \\ P(n) \leqslant t}} \left(\frac{d}{n}\right) n^{-s}$$

$$= O\Big(\sum_{\substack{n \geqslant x^{2/3} \\ P(t) \leqslant t}} n^{-s}\Big), \tag{4}$$

since the number of integers $n \leqslant x$ for which $P(n) \leqslant t$ is less than

$$\left(\frac{\log x}{\log 2}\right)^{\pi(t)} < \left(\frac{\log x}{\log 2}\right)^t,$$

where $\pi(y)$ denotes the number of primes $\leqslant y$. We obtain by partial summation from (4) that

$$|L_d^{(t)}(s)-L_d^{(t)}(s, x^{\frac{2}{3}})| < c \left(\frac{\log x}{\log 2}\right)^t x^{-\frac{2}{3}s} = o(1), \qquad (5)$$

as x tends to ∞, for any fixed t.

Combining Lemma 2 with (3) and (5) we obtain

LEMMA 3. *For every positive δ there exist an ε and t_0 so that for the number of integers d with $1 < d \leqslant x$ $[d \equiv 0, 1 \ (4), d \neq u^2]$ satisfying*

$$|L_d^{(t)}(s) - L_d(s)| > \varepsilon$$

is $\leqslant \delta x$ whenever $t > t_0$ and x is large enough $[x > x_0(\delta)]$.

5. We define the primes

$$p_1 < p_2 < p_3 \ldots\ldots < p_k$$

and all less than t as follows:

$p_1 = 5$; p_{i+1} is the least prime satisfying

$$p_{i+1}^s > 10\, p_i^s \quad (1 \leqslant i \leqslant k-1).$$

We define the signature of d with respect to a set of primes as the set of values of (d/p), where p runs through the given set of primes.

Denote by $q_1, q_2, \cdots, q_m$ the primes $\leqslant t$ which are distinct from the p's. We have $k+m = \pi(t)$.

Let d_1 and d_2 be two values of d which have different signatures with respect to the p's but the same signature with respect to the q's. Let p_j $(j \leqslant k)$ be the first prime p for which the signatures of d_1 and d_2 disagree. Then we clearly have

$$\text{Max}\left(\frac{L_{d_1}^{(t)}(s)}{L_{d_2}^{(t)}(s)}, \frac{L_{d_2}^{(t)}(s)}{L_{d_1}^{(t)}(s)}\right) \geqslant (1+p_j^{-s}) \prod_{t=j+1}^{k} \left(\frac{1-p_t^{-s}}{1+p_t^{-s}}\right)$$

$$> (1+p_j^{-s}) \prod_{t=j+1}^{k} (1-2p_t^{-s}) > (1+p_j^{-s}) \left\{1-2\sum_{t=j+1}^{k} p_t^{-s}\right\}$$

$$> (1+p_j^{-s}) \left\{1-\frac{2}{p_j^s}\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\cdots\right)\right\}$$

$$= (1+p_j^{-s})\left(1-\frac{2}{9}p_j^{-s}\right)$$

$$= 1+\tfrac{7}{9}p_j^{-s}-\tfrac{2}{9}p_j^{-2s} > 1+\tfrac{5}{9}p_j^{-s} > 1+\tfrac{1}{2}p_j^{-s}$$
$$\geqslant 1+\tfrac{1}{2}p_k^{-s}. \qquad (6)$$

In the above we used that $2p_j^{-s} < 1$ which follows from $s > \frac{3}{4}$, $p_1 = 5$. Also we used $p_{i+1}^s > 10\, p_i^s$. We next prove

LEMMA 4. *If* $a > 0$ *and* $0 < \varepsilon < \dfrac{a}{4\, p_k^s + 1}$ *the inequality*

$$a-\varepsilon < L_d^{(t)}(s) < a+\varepsilon$$

cannot be satisfied by $d = d_1$ *and* $d = d_2$ *if* d_1 *and* d_2 *are values of* d *such that their signature with respect to the primes* p *is different, while their signature with respect to the primes* q *is the same.*

From (6) it follows that Lemma 4 is true if ε is so small that

$$\frac{a+\varepsilon}{a-\varepsilon} < 1+\frac{1}{2\, p_k^s},$$
$$\frac{\varepsilon}{a} < \frac{2^{-1}\, p_k^{-s}}{2+2^{-1}\, p_k^{-s}},$$
$$\varepsilon < \frac{a}{1+4\, p_k^s}.$$

This proves the lemma.

Let y_s denote the number of $d \leqslant x$ [$d > 0$; $d \equiv 0, 1\ (4)$; $d \neq u^2$] such that all the d's have a fixed signature with respect to the q's. Clearly s assumes

$$h = 3^m = 3^{\pi(t)-k}$$

values, and

$$y_1 + y_2 + \cdots + y_h = \frac{x}{2} + O(\sqrt{x}),$$

where the constant implied in the O is an absolute one.

Again the d's (which have a fixed signature with respect to the q's) fall into $3^k = g$ classes according to their signature with respect to the p's. Thus

$$y_s = z_{1s} + z_{2s} + \cdots + z_{gs} \quad (1 \leqslant s \leqslant h). \qquad (7)$$

Clearly

$$gh = 3^{\pi(t)}.$$

Next we prove

LEMMA 5. *For* $x > x_0(k)$ *we have*

$$\frac{z_{bs}}{y_s} < \frac{1}{2^k}\ [1 \leqslant b \leqslant g].$$

PROOF. The d's $\leqslant x$ which have a particular signature with respect to the q's are (by assumption) y_s in number ($s = 1, 2, 3, \dots, h$). Let

$$q_{\alpha_1}, q_{\alpha_2}, \dots, q_{\alpha_w}$$

be the primes q for which $(d/q) = 0$; let

$$q_{\beta_1}, q_{\beta_2}, \dots, q_{\beta_{w'}}$$

be the primes for whih $(d/q) = +1$; finally let

$$q_{\gamma_1}, q_{\gamma_2}, \dots, q_{\gamma_{w''}}$$

be the primes q for which $(d/q) = -1$. We have

$$w + w' + w'' = \pi(t) - k = m.$$

It is evident that

$$y_s = \frac{x}{2} \prod_{n=1}^{w} q_{\alpha_n}^{-1} \prod_{n=1}^{w'} \left(\frac{q_{\beta_n} - 1}{2\, q_{\beta_n}}\right) \prod_{n=1}^{w''} \left(\frac{q_{\gamma_n} - 1}{2\, q_{\gamma_n}}\right) + O(\sqrt{x})$$

$$= \frac{Qx}{2} + O(\sqrt{x}), \tag{8}$$

where the constant in the last O may also depend on t. Consider next the value of z_{bs}. This number is the number of $d \leqslant x$ which have the above signature with respect to the q's and also have a fixed signature with respect to the p's. Write

$$\left(\frac{d}{p}\right) = 0 \quad \text{for } p = p_{\alpha_n}\ (1 \leqslant n \leqslant v).$$

$$\left(\frac{d}{p}\right) = +1 \quad \text{for } p = p_{\beta_n}\ (1 \leqslant n \leqslant v').$$

$$\left(\frac{d}{p}\right) = -1 \quad \text{for } p = p_{\gamma_n}\ (1 \leqslant n \leqslant v'').$$

Then

$$v + v' + v'' = k.$$

Further it is evident that

$$z_{bs} = \frac{Qx}{2} \prod_{n=1}^{v} p_{\alpha_n}^{-1} \prod_{n=1}^{v'} \left(\frac{p_{\beta_n} - 1}{2 p_{\beta_n}}\right) \prod_{n=1}^{v''} \left(\frac{p_{\gamma_n} - 1}{2 p_{\gamma_n}}\right)$$

$$= PQ\frac{x}{2} + O(\sqrt{x}). \tag{9}$$

From (8) and (9), we have

$$z_{bs}/y_s = P + O(x^{-\frac{1}{2}}). \tag{10}$$

The lemma thus follows since $P < 2^{-k}$ $[k > 1]$.

Consider now the d's which have the same signature as the numbers of y_s $(1 \leqslant s \leqslant h)$. By Lemma 4 at most max (z_{bs}) $[1 \leqslant b \leqslant 3^k]$ of them satisfy the inequality

$$a - \varepsilon < L_d^{(t)}(s) < a + \varepsilon. \tag{11}$$

Hence by Lemma 5 the total number of d's not exceeding x which satisfy (11) is at most

$$z_{\alpha 1} + z_{\beta 2} + z_{\gamma 3} + \cdots < 2^{-k}(y_1 + y_2 + y_3 + \cdots) = 2^{-k}\left\{\frac{x}{2} + O(\sqrt{x})\right\}.$$

Thus, we have (choose $2^{-k} \leqslant \delta$)

LEMMA 6. *Given any positive δ, there exist t_0, ε, x_0 such that the number of positive $d \leqslant x$ with $d \equiv 0, 1 \ (4)$, $d \neq u^2$, and*

$$a - \varepsilon < L_d^{(t)}(s) < a + \varepsilon \tag{12}$$

is less than δx for all $t > t_0$, $x > x_0$.

The case $a = 0$ needs special discussion. Here (12) has to be replaced by

$$0 < L_d^{(t)}(s) < \varepsilon \tag{13}$$

whence

$$S_d = \prod_{p \leqslant t} \left\{1 - \left(d/p\right) p^{-s}\right\} > \varepsilon^{-1}.$$

Now the sum

$$\sum_{1 \leqslant d \leqslant x} S_d = x/2 + O(\sqrt{x}),$$

where d runs over integers which are $\equiv 0, 1 \ (4)$, not perfect squares, and $\leqslant x$.

It easily follows that Lemma 6 is true with $a = 0$ when we replace (12) by (13).

6. **Proof of the theorem.** Denote by $g_t(a, x)$ the number of integers $d \leqslant x$ $[d \equiv 0, 1 \ (4), d \neq u^2, d > 0]$ for which

$$L_d^{(t)}(s) \leqslant a.$$

It is easy to see that

$$\lim_{x\to\infty} \frac{g_t(a, x)}{x/2} = g(a, t)$$

exists. This follows from the following simple observation. The expression

$$\prod_{p\leq t} \left\{ 1-\left(\frac{d}{p}\right)p^{-s}\right\}^{-1}$$

is periodic in d (mod $p_1 p_2 \cdots p_w$), where $p_1, \ldots, p_w$ are all the primes $\leq t$. As d goes from 1 to $p_1 p_2 \cdots p_w$ suppose that there are N_t values of d for which

$$0 < L_d^{(t)}(s) \leq a.$$

Then

$$\lim_{x\to\infty} \frac{g_t(a, x)}{x/2} = \frac{2N_t}{p_1 p_2 \cdots p_w}.$$

We next prove

$$\lim_{t\to\infty} g(a, t) = g(a),$$

where $g(a)$ was defined in § 1.

To do this it will suffice to show that given an arbitrary positive η we can find t_0, x_0 such that

$$|g_t(a, x) - g(a, x)| < \eta x$$

for $t > t_0$, $x > x_0(\eta)$.

We split the integers $d \leq x$ [$d \equiv 0, 1$ (4), $d \neq u^2$] which satisfy

$$L_d^{(t)}(s) \leq a,\ L_d(s) > a$$

or

$$L_n^{(t)}(s) > a,\ L_d(s) \leq a$$

into two classes:

I $$|L_d^{(t)}(s) - L_d(s)| > \varepsilon.$$

By Lemma 3 the number of these integers is $< \delta x$.

II $$a-\varepsilon < L_d^{(t)}(s) < a+\varepsilon.$$

By Lemma 6 the number of these integers is $< \delta x$.

This completes the proof of our theorem. The fact that $g(a)$ is a continuous and strictly increasing function of a follows easily by the arguments of Lemmas 3 and 6.

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ON SEQUENCES OF POSITIVE INTEGERS

By H. DAVENPORT AND P. ERDÖS

Let $a_1, a_2, \ldots, a_m$ be any finite set of distinct natural numbers, and let $b_1, b_2, \ldots$ be the sequence formed by all those numbers which are divisible by any of $a_1, a_2, \ldots, a_m$. This sequence has a density in the obvious sense and we denote this density by $A(a_1, a_2, \ldots, a_m)$. In fact

$$A(a_1, a_2, \ldots, a_m) = \frac{1}{a_1} + \left(\frac{1}{a_2} - \frac{1}{[a_1, a_2]}\right) + \left(\frac{1}{a_3} - \frac{1}{[a_1, a_3]} - \frac{1}{[a_2, a_3]} + \frac{1}{[a_1, a_2, a_3]}\right) + \cdots, \quad (1)$$

where $[a, b, \ldots]$ denotes the least common multiple of a, $b, \ldots$. For the first term above represents the density of the multiples of a_1, the second represents the density of those multiples of a_2 that are not multiples of a_1, and so on.

Now suppose we start from an infinite sequence a_1, $a_2, \ldots$ (arranged in increasing order) instead of from a finite set. It is plain that $A(a_1, a_2, \ldots, a_m)$ increases with m, and is always less than 1. We define

$$A = \lim_{m\to\infty} A(a_1, a_2, \ldots, a_m). \quad (2)$$

It is natural to expect that A should again be the density, in some sense, of the sequence $b_1, b_2, \ldots$ formed by all numbers which are divisible by any of $a_1, a_2, \ldots$. This cannot be true for the ordinary density, since it was proved by Besicovitch [1] that the b sequence may have different upper and lower densities.

There is one specially simple case in which the conclusion does hold, namely when the series $\Sigma\, 1/a_n$ converges. For, in this case, the number of b's up to x which are not divisible by any of $a_1, a_2, \ldots, a_m$ is at most

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$$\left[\frac{x}{a_{m+1}}\right]+\left[\frac{x}{a_{m+2}}\right]+\cdots;$$

and this after division by x is less than $\sum_{n=m+1}^{\infty} 1/a_n$. Hence the proportion of numbers that are b's differs from $A(a_1, a_2, \ldots, a_m)$ by an amount which tends to zero as $m \to \infty$, and the conclusion follows. Under these conditions, we can safely use the notation $A(a_1, a_2, \ldots)$ for A.

We proved some years ago [2] that, in the general case, the number A represents the *lower* density of the b sequence. We also proved that the b sequence has a *logarithmic density*, and that this also equals A. The logarithmic density is defined as

$$\lim_{x\to\infty} \frac{\beta(x)}{\log x},$$

where

$$\beta(x) = \sum_{b_i \leqslant x} \frac{1}{b_i}. \tag{3}$$

The proof of these two results was somewhat indirect, since it used Dirichlet series and appealed to a Tauberian theorem of Hardy and Littlewood. Our object in the present note is to give a direct and elementary proof.

Let us denote the upper and lower densities of the b sequence by d and D and the upper and lower logarithmic densities by δ and Δ. It is well known that

$$d \leqslant \delta \leqslant \Delta \leqslant D$$

for any sequence. In the present case, it is immediate that $d \geqslant A$. For the b sequence includes the multiples of $a_1, a_2, \ldots, a_m$, and so $d \geqslant A(a_1, a_2, \ldots, a_m)$ for each m, whence $D \geqslant A$. To complete the proof of the two results just enunciated, it suffices to prove that $\Delta \leqslant A$, in other words that

$$\overline{\lim_{x\to\infty}} \frac{\beta(x)}{\log x} \leqslant A. \tag{4}$$

The new proof of (4) is based on the consideration of what may be called the *multiplicative density* of a sequence. Let $p_1, p_2, \ldots, p_k$ be the first k primes. Denote by n' the general number composed entirely of these primes; then $\Sigma\, 1/n'$ converges, and

$$\Sigma \frac{1}{n'} = \prod_{i=1}^{k} \left(1 - 1/p_i\right)^{-1} = \Pi_k, \text{ say.} \tag{5}$$

Now denote by $b_1', b_2', \ldots$ those numbers of the b sequence that are composed entirely of $p_1, p_2, \ldots, p_k$. Let

$$B_k = \frac{\Sigma\, 1/b_i'}{\Sigma\, 1/n'} = (\Pi_k)^{-1} \Sigma\, 1/b_i'. \tag{6}$$

This fraction may be said to measure the density of the numbers b' among the numbers n'. If B_k tends to a limit as $k \to \infty$, we may call this limit the multiplicative density of the b sequence.

In the case under consideration here, where the b sequence consists of all multiples of $a_1, a_2, \ldots$, we can easily prove that the multiplicative density exists and has the value A. Let us denote by $a_1', a_2', \ldots$ those a's which are composed entirely of the primes $p_1, p_2, \ldots, p_k$. Then the b' consists of all numbers of the form $a'\, n'$, but without repetition. Hence we have

$$\Sigma \frac{1}{b'} = \frac{1}{a_1'} \Sigma \frac{1}{n'} + \left(\frac{1}{a_2'} - \frac{1}{[a_1', a_2']}\right) \Sigma \frac{1}{n'} + \ldots = \Pi_k A(a_1'\, a_2', \ldots).$$

It follows that

$$B_k = A(a_1', a_2', \ldots). \tag{7}$$

By an earlier remark, since $\Sigma\, 1/a'$ is convergent, we know that this is the density, in the ordinary sense, of the sequence formed by all multiples of $a_1', a_2', \ldots$. It is plain from (7) that B_k increases with k, and is always less than 1. Hence

$$B = \lim_{k \to \infty} B_k$$

exists, and our next step is to prove that

$$B = A. \tag{8}$$

In the first place, if k is sufficiently large in relation to m, the numbers $a_1', a_2', \ldots$ include $a_1, a_2, \ldots, a_m$. Hence $B \geqslant A(a_1', a_2' \ldots) \geqslant A(a_1, a_2, \ldots, a_m)$, whence $B \geqslant A$. Next, since $\Sigma\, 1/a'$ converges, we have for fixed k (by an argument used earlier)

$$A(a_1', a_2', \ldots) \leqslant A(a_1', a_2', \ldots, a_m') + \sum_{n=m+1}^{\infty} \frac{1}{a_n'}.$$

Now, if we choose r so large that $a_1', a_2', \ldots, a_m'$ are all included in $a_1, a_2, \ldots, a_r$, we have

$$A(a_1', a_2', \ldots, a_m') \leqslant A(a_1, a_2, \ldots, a_r) \leqslant A.$$

Making $m \to \infty$, we obtain

$$A(a_1', a_2', \ldots) \leqslant A,$$

that is, $B_k \leqslant A$. Hence $B \leqslant A$, which proves (8).

After this preparation, we proceed to prove (4). We divide the numbers $b_i \leqslant x$ into two classes, placing in the first class those divisible by any of $a_1', a_2', \ldots$. Here $a_1', a_2', \ldots$ are again those a's that are composed entirely of $p_1, p_2, \ldots, p_k$. For fixed k, the b's in the first class have density B_k, by (7). Hence the sum $\beta_1(x)$ corresponding to the b's in the first class satisfies

$$\lim_{x \to \infty} \frac{\beta_1(x)}{\log x} = B_k. \tag{9}$$

To estimate the sum $\beta_2(x)$ corresponding to the b's in the second class, we introduce a prime p_h defined by $p_h \leqslant x < p_{h+1}$. The b's in the second class are composed entirely of $p_1, p_2, \ldots, p_h$, but are not divisible by any a composed entirely of $p_1, p_2, \ldots, p_k$. If we denote by b^* the b's of this kind, whether less than x or not, we have

$$\beta_2(x) \leqslant \Sigma\, 1/b^*. \tag{10}$$

The numbers b^* can be obtained by taking all numbers b composed entirely of $p_1, p_2, \ldots, p_h$, say all numbers b'', and removing from them all numbers $b' t$, where b' is composed entirely of $p_1, p_2, \cdots, p_k$ and t is any number composed entirely of $p_{k+1}, p_{k+2}, \cdots, p_h$. Hence

$$\Sigma 1/b^* = 1/b'' - (\Sigma 1/b')\,(\Sigma 1/t) = \Pi_h\, B_h - \Pi_k\, B_k\, \Sigma 1/t,$$

by two appeals to (6). Since

$$\sum \frac{1}{t} = \prod_{i=k+1}^{h} \left(1 - \frac{1}{p_i}\right)^{-1} = \Pi_h\, (\Pi_k)^{-1},$$

we have

$$\Sigma 1/b^* = \Pi_h\, (B_h - B_k). \tag{11}$$

Now it is well known that Π_h, defined by (5) with h in place of k, satisfies [3, p. 22]

$$\Pi_h < C \log p_h \leqslant C \log x,$$

where C is an absolute constant. Hence, by (10) and (11), we have

$$\beta_2(x) < C(B_h - B_k) \log x. \tag{12}$$

It follows from (9) and (12) that

$$\overline{\lim_{x\to\infty}} \frac{\beta(x)}{\log x} \leqslant B_k + C(B - B_k).$$

Since $B_k \to B$ as $k \to \infty$, this proves (4).

It may be observed, in conclusion, that the density taken in any sense which is essentially stronger than the logarithmic sense, need not exist. For example, if $\alpha < 1$, the density in the sense of

$$\lim_{x\to\infty} (1-\alpha)\, x^{\alpha-1} \sum_{b_i \leqslant x} 1/b_i^{\alpha}$$

need not exist. This follows from the example constructed by Besicovitch. If a function of b_i is used which increases very little less rapidly than b_i, for instance $b_i/(\log b_i)$, the density will exist in the sense that

$$2\,(\log x)^{-2} \sum_{b_i \leqslant x} (\log b_i)/b_i \to A.$$

But this is at once seen to be equivalent to $\beta(x)/(\log x) \to A$, on applying partial summation; so that nothing essentially new is obtained.

Note added May 1951. It may be of interest to observe that results similar to those proved above about that b sequence can sometimes be proved for the sequence formed by those b's which satisfy a supplementary condition. Consider, for example, those b_i for which $b_{i+1}-b_i=k$, where k is a given positive integer. It can be proved that these b_i have a logarithmic density; and that they have a density in the ordinary sense, provided that the whole b sequence has a density. The method of proof is to start from the case of a finite set $a_1, a_2, \ldots, a_m$, in which case the b's form a periodic sequence.

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THE ASYMPTOTIC BEHAVIOUR OF A FUNCTION OCCURRING IN THE THEORY OF PRIMES

By N. G. DE BRUIJN

1. Introduction. During the last few years several mathematicians (S. D. Chowla and T. Vijayaraghavan [4], V. Ramaswami[5], A. A. Buchstab[1] and others*) discussed the number of integers which are $\leqslant x$ and free of prime divisors $>y$. We shall denote the number by $\Psi(x, y)$.

The problem of the asymptotic behaviour, for x and y tending independently to infinity, seems to be difficult, [5, p. 101]. However, a part of it can be solved completely. If u is a fixed positive number, and if $x=y^u$, then we know [4, 1] that

$$\lim_{y\to\infty} y^{-u}\Psi(y^u, y) = \rho(u), \qquad (1.1)$$

where $\rho(u)$ is continuous for $u>0$, and uniquely defined by

$$\rho(u) = 1 \quad (0 \leqslant u \leqslant 1), \quad u\rho'(u) = -\rho(u-1) \quad (u>1). \qquad (1.2)$$

The function $\rho(u)$ is positive, and decreases rather rapidly as $u \to \infty$. Ramaswami [5] proved that

$$C_1 u^{-1} 4^{-u} \{\Gamma(u)\}^{-2} < \rho(u) < C_2\{u\Gamma(u)\}^{-1} \quad (u \geqslant 1) \qquad (1.3)$$

for suitable constants C_1 and C_2. The first inequality was improved by Buchstab[1], who proved that

$$\rho(u) > \exp\left(-u\{\log u + \log\log u + 6(\log\log u)/(\log u)\}\right),$$
$$(u \geqslant 6). \qquad (1.4)$$

In the present paper we shall prove that the asymptotic behaviour of $\rho(u)$ is given by (1.6).

Let, for $u>1$, ξ be the positive root of the equation

$$e^{\xi} - 1 = u\xi. \qquad (1.5)$$

* The late S. S. Pillai was one of them, see [4, p. 35] and [5, p. 101]. It seems, however, that he did not publish anything about the problem himself.

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Then we have the asymptotic formula

$$\rho(u) \sim \frac{e^{\gamma}}{\sqrt{(2\pi u)}} \exp\left\{-\int_0^{\xi} \frac{se^s - e^s + 1}{s}\, ds\right\} \quad (u \to \infty), \qquad (1.6)$$

where γ is Euler's constant. A proof will be given in §2.

Writing $(e^s - 1)/s = \eta$, the integral in (1.6) can be transformed as follows

$$\int_0^{\xi} = \int_1^{u} s\, d\eta. \qquad (1.7)$$

Using $s = \log(1+\eta s) = \log\{1+\eta(\log(1+\eta s))\}$, etc. it is not difficult to give a rough estimate of (1.7) in terms of elementary functions. So we obtain, for instance

$$\rho(u) = \exp\left[-u\left\{\log u + \log\log u - 1 - \frac{1}{\log u} + \frac{\log\log u}{\log u} + O\left(\frac{(\log\log u)^2}{(\log u)^2}\right)\right\}\right]. \qquad (1.8)$$

In §3 we shall prove the following exact formula for $\rho(u)$.

$$\rho(u) = \frac{e^{\gamma}}{2\pi i}\int_{-i\infty}^{i\infty} \exp\left\{-uz + \int_0^{z} \frac{e^s - 1}{s}\, ds\right\} dz, \quad (u > 0). \qquad (1.9)$$

If $u < 0$, the right-hand-side vanishes identically. For $u = 0$ the integral is not convergent in the ordinary sense.

For several reasons we did not use this formula for the proof of (1.6) in §2. First, the saddle-point analysis of (1.9) is in some respects more difficult than that of (2.4). Secondly, the method evolved in §2 may be applicable to other problems also. For instance, it immediately applies to functions $f(u)$ given arbitrarily for $0 \leqslant u \leqslant 1$ and continued by $uf'(u) = -f(u-1)$ $(u > 1)$. On the other hand, the proof in §2 has the disadvantage of borrowing from a preceding paper[3] a fact which was rather difficult to prove.

The problem of $\Psi(x, y)$, for x and y tending independently to infinity is in many respects analogous to the one of $\Phi(x, y)$, i.e. the number of integers $\leqslant x$ which are free of prime factors $\leqslant y$. This function $\Phi(x, y)$ was studied

in a previous paper[2], and the methods used there lead to some improvements on Ramaswami's results[5] when applied to $\psi(x, y)$. This will be the subject of a forthcoming paper [6].

2. **Proof of the asymptotic formula for $\rho(u)$.** We shall determine the asymptotic behaviour of the function $\rho(u)$, defined by (1.2). We define $\rho(u) = 0$ for $u < 0$. Then we have

$$u\rho(u) = \int_0^1 \rho(u-t)\,dt \qquad (-\infty < u < \infty). \qquad (2.1)$$

For, both members of (2.1) have, for $u > 1$, the same derivative in virtue of (1.2), and (2.1) is trivial for $u \leqslant 1$.

The equation

$$uF(u) = \int_0^1 F(u-t)\,dt \qquad (2.2)$$

was considered as an example in a previous paper[3]. There we proved the following:

Let both $F(u)$ and $F_0(u)$ be continuous for $u > 0$, both satisfying (2.2) for $u > 0$. Assume $F_0(u)$ to be positive and non-increasing for $u \geqslant 1$. Then we have $F(u) = \{ C + O(u^{-1}) \} F_0(u)$, where C is a constant.

Our function $\rho(u)$ satisfies the conditions imposed on $F_0(u)$, and so we have

$$F(u) = \{ C + O(u^{-\frac{1}{2}}) \} \rho(u). \qquad (2.3)$$

We shall now construct a second solution $F_1(u)$ of (2.2), whose asymptotic behaviour for $u \to \infty$ can be determined. We use Laplace's method, i.e. we try to find a solution in the form $F(u) = \int e^{uz} p(z)\,dz$, taken along some path in the complex z-plane. From formal calculations we obtain

$$p(z) = \exp\left\{ \int_0^z \frac{e^s - 1}{s}\,ds \right\}.$$

This function $p(z)$ tends to 0 very rapidly as z tends to infinity along the lines $z = \pm \pi i + t$, $t \to \infty$. Thus we take

$$F_1(u) = \frac{1}{2\pi i}\int_W \exp\left\{ -uz + \int_0^z \frac{e^s - 1}{s}\,ds \right\} dz, \qquad (2.4)$$

where W is the path consisting of the three segments $(-\pi i+\infty, -\pi i)$, $(-\pi i, \pi i)$, $(\pi i, \pi i+\infty)$. Clearly $F_1(u)$ is an entire function which is real for real values of u.

The asymptotic behaviour, for $u \to \infty$, of $F_1(u)$ can be determined by the saddle-point method. Omitting the details, we only give a brief sketch.

The integrand of (2.4) is denoted by $\exp\{\psi(z)\}$. The saddle-points are the zeros of the function $\psi'(z) = -u+(e^z-1)/z$. For u sufficiently large, $\psi(z)$ has only one zero $\xi(u)$ in the strip $|Im\, z| \leqslant \pi$, and we have $\xi(u) \to \infty$ as $u \to \infty$. Replacing the path W by a path through this saddle-point, the value of $F(u)$ can be approximated. We obtain, for $u \to \infty$,

$$F_1(u) \sim \frac{e^{\psi(\xi)}}{2\pi}\left(\frac{2\pi}{\psi''(\xi)}\right)^{\frac{1}{2}}, \tag{2.5}$$

where ξ is the positive root of $e^\xi - 1 = u\xi$. For $\psi''(\xi)$ we obtain $\{(\xi-1)e^\xi-1\}\xi^{-2}$; using $\psi'(\xi) = 0$ we deduce

$$\psi''(\xi) = u-(u-1)\xi^{-1} \sim u. \tag{2.6}$$

In order to determine the constant C in (2.3) we use a method applied in a previous paper (see[2], §4) to a similar problem. The present case is more difficult, a second application of the saddle-point method being necessary.

Let $F(u)$ be a solution of (2.4), or, more generally a solution of $uF'(u) = -F(u-1)$, for $u > 0$. Anyhow we require that $F(u)$ is continuous for $u > 0$ and integrable for $u > -1$. Further, let $G(u)$ satisfy, for $u > 0$, the adjoint equation

$$uG'(u-1) = G(u)-G(u-1), \tag{2.7}$$

then the expression

$$(F, G) = \int_{a-1}^{a} F(u)G(u)du - aF(a)G(a-1) \tag{2.8}$$

does not depend on a for $a > 0$. This is easily verified by differentiation with respect to a.

Again by Laplace's method we construct a special solution G_1 of (2.7):

$$G_1(u) = V.P. \int_{-\infty}^{\infty} \exp\left\{ uz - \int_0^z \frac{e^s - 1}{s}\, ds \right\} . \frac{e^z}{z} dz \ (u > -1). \tag{2.9}$$

Here $V.P.$ denotes the principal value:

$$V.P. \int_{-\infty}^{\infty} = \lim_{\delta \to 0} \left\{ \int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \right\}.$$

The integral (2.9) converges for $u > -1$, and it is not difficult to verify that $G_1(u)$ satisfies (2.7) for $u > 0$.

$G(u)$ tends to infinity as $u \to \infty$, and we shall determine its asymptotic behaviour.

The saddle-point of the integrand is very close to ξ, the most important factor being $\exp\{-\psi'z)\}$, where $\psi(z)$ is the same function as above. We easily deduce, for $u \to \infty$,

$$G_1(u) \sim e^{-\psi(\xi)} \left(\frac{2\pi}{\psi''(\xi)} \right)^{\frac{1}{2}} \frac{e^\xi}{\xi}, \tag{2.10}$$

$$G_1(u-1) \sim e^{-\psi(\xi)} \left(\frac{2\pi}{\psi''(\xi)} \right)^{\frac{1}{2}} \cdot \frac{1}{\xi}, \tag{2.11}$$

where ξ is, again, the positive solution of $e^\xi - 1 = u\xi$. It now follows from (2.5) and (2.6) that

$$F_1(u)\, G_1(u) \sim \frac{e^\xi}{\xi\psi''(\xi)} \sim 1, \qquad (u \to \infty), \tag{2.12}$$

$$uF_1(u)\, G_1(u-1) \sim \frac{u}{\xi\psi''(\xi)} \sim \frac{1}{\xi}, \quad (u \to \infty). \tag{2.13}$$

Therefore, using the fact that (2.8) does not depend on a, we make $a \to \infty$, and obtain

$$(F_1, G_1) = 1. \tag{2.14}$$

On the other hand we can determine (ρ, G_1) from the fact that $\rho(u) = 0\ (-1 < u < 0)$, $\rho(u) = 1\ (0 \leqslant u \leqslant 1)$. We obtain from (2.8) making $a \downarrow 0$,

$$(\rho, G_1) = \lim_{a \downarrow 0} \left\{ -aG_1(a-1) \right\}.$$

It is easily seen from (2.9) that $V.P. \int_{-1}^{\infty}$ is bounded for

$a \downarrow 0$. Hence we only have to deal with $\int_{-\infty}^{-1}$. It follows that

$$\lim_{a \downarrow 0} \{-aG_1(a-1)\} = \lim_{a \downarrow 0} a \int_1^\infty e^{-az} \varphi(z)\, dz, \qquad (2.15)$$

where

$$\varphi(z) = z^{-1} \exp\left\{-\int_0^{-z} \frac{e^s - 1}{s}\, ds\right\}. \qquad (2.16)$$

We have $\varphi(z) \to e^\gamma$ as $z \to +\infty$, where γ is Euler's constant. It follows that the limit in (2.15) equals e^γ, and so

$$(\rho, G_1) = e^\gamma. \qquad (2.17)$$

We know from (2.3) that $F_1(u)/\rho(u)$ tends to a limit C as $u \to \infty$, and so $F_1 - C\rho = o(\rho)$. Therefore, we obtain from (2.8), (2.12) and (2.13) by making $a \to \infty$, that $(F_1 - C\rho, G_1) = 0$. It follows, by (2.14) and (2.17), that $C = e^{-\gamma}$. Now (1.6) follows from (2.5) and (2.6).

If we would be satisfied with the fact that C is positive, a simpler proof is possible. For it follows from (2.5) that $F_1(u)$ has a positive lower bound in some interval $A \leqslant u \leqslant A+1$. Consequently we have, for some $c > 0$, $F_1(u) - c\rho(u) \geqslant 0$ $(A \leqslant u \leqslant A+1)$. But $F_1(u) - c\rho(u)$ satisfies (2.2) for $u > 0$, and it easily follows that it is > 0 for all $u \geqslant A$. This means that $C \geqslant c$.

3. **Proof of the exact formula for $\rho(u)$.** We shall show that the following integral, taken along the imaginary axis,

$$R(u) = \frac{e^\gamma}{2\pi i} \int_{-i\infty}^{i\infty} \exp\left\{-uz + \int_0^z \frac{e^s - 1}{s}\, ds\right\} dz, \qquad (3.1)$$

equals $\rho(u)$ for $u \neq 0$; we again put $\rho(u) = 0$ for $u < 0$. $R(u)$ is meaningless for $u = 0$.

We first show the convergence for $u > 0$ and $u < 0$.

For $Re\, z \leqslant 0$ we can write (cf. (2.16))

$$\exp\left\{\gamma + \int_0^z \frac{e^s - 1}{s}\, ds\right\} = -z^{-1} \exp\left\{\int_{-\infty}^{z} \frac{e^s}{s}\, ds\right\}. \qquad (3.2)$$

For $Re\, z \leqslant 0$ we have furthermore

$$\left| \int_{-\infty}^{z} \frac{e^s}{s}\, ds \right| \leqslant \int_{z-\infty}^{z} \left| \frac{e^s}{z} \right| ds = \left| \frac{e^z}{z} \right|,$$

and so, if we write

$$e^{\gamma} \exp\left\{ \int_0^z \frac{e^s - 1}{s}\, ds \right\} = -\frac{1}{z} + K(z), \tag{3.3}$$

then we have

$$K(z) = O(|z|^{-2}) \qquad (Re\, z \leqslant 0, |z| \geqslant 1). \tag{3.4}$$

It follows that (3.1) is convergent for $u > 0$ and for $u < 0$.

We have $R(u) = 0$ for $u < 0$. For, we may shift the path of integration over an arbitrary distance A to the left. Then we find, by (3.3) and (3.4)

$$2\pi i R(u) = -\int_{-A-i\infty}^{-A+i\infty} \frac{e^{-uz}}{z}\, dz + O\left\{ \int_{-A-i\infty}^{-A+i\infty} \frac{dz}{|z|^2} \right\} = O(A^{-1}),$$

since the first integral vanishes for $u < 0$. Hence $R(u) = 0$.

Furthermore, we have

$$R(u) = \frac{-1}{2\pi i} \int_{-1-i\infty}^{-1+i\infty} \frac{e^{-uz}}{z}\, dz + \frac{1}{2\pi i} \int_{-1-i\infty}^{-1+i\infty} e^{-uz} K(z)\, dz \quad (u \neq 0), \tag{3.5}$$

and $K(z)$ is regular and $O(|z|^{-2})$ for $Re\, z = -1$. Therefore, the second integral is a continuous function of the real variable u. The first term, however, is 0 for $u < 0$, and 1 for $u > 0$. It follows that $R(u)$ has a jump 1 for $u = 0$, and so

$$R(u) \to 1 \qquad (u > 0, u \to 0). \tag{3.6}$$

Furthermore we infer from (3.5) that $R(u)$ is continuous for $u > 0$.

We now determine $\int_0^1 R(u-t)\, dt$, and we carry out the integration with respect to t in both terms of (3.5). The second term is easy. By (3.4) we have

$$\int_0^1 dt \int_{-1-i\infty}^{-1+i\infty} e^{-(u-t)z} K(z) dz = \int_{-1-i\infty}^{-1+i\infty} e^{-uz} K(z) \frac{e^z - 1}{z} dz$$
$$(-\infty < u < \infty). \tag{3.7}$$

Denoting the first term of the right-hand-side of (3.5) by $R_1(u)$, we can show also that

$$\int_0^1 R_1(u-t)\,dt = \frac{-1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} \frac{e^{-uz}}{z}\cdot\frac{e^z-1}{z}\,dz \quad (-\infty < u < \infty). \tag{3.8}$$

For, since $R_1(u)$ equals 0 for $u<0$ and 1 for $u>0$, the left-hand-side is 0 for $u<0$, u for $0 \leqslant u \leqslant 1$ and 1 for $u \geqslant 1$. It is not difficult to obtain the same result for the right-hand-side, by the theorem of residues.

By (3.7) and (3.8) we have, for all values of u,

$$\int_0^1 R(u-t)\,dt = \frac{e^\gamma}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} \exp\left\{-uz+\int_0^z \frac{e^s-1}{s}ds\right\}\cdot\frac{e^z-1}{z}\,dz.$$

In virtue of (3.3) and (3.4) the following partial integration is legitimate:

$$\int_0^1 R(u-t)\,dt = -\frac{e^\gamma}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} \exp\left\{\int_0^z \frac{e^s-1}{s}ds\right\}\cdot\frac{d}{dz}e^{-uz}\,dz.$$

Clearly, this is 0 if $u=0$, and it equals $u\,R(u)$ otherwise. Therefore, $R(u)$ satisfies the equation (2.2) for all u, and, since $R(u)=0$ for $u<0$, and (3.6) holds, we easily infer that $R(u)=\rho(u)$ for $u>0$ also. This proves (1.9).

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ON THE GENERATING FUNCTION OF THE INTEGERS WITH A MISSING DIGIT

By K. Mahler

Let n be a positive integer such that no digit in its decimal representation is equal to zero, and let N be the set of all such integers n. It is well known that the series

$$\sigma = \sum_{n \in N} 1/n$$

converges. Whether its value σ is a transcendental number, or whether it can be expressed by means of elementary transcendental functions, is, however, a difficult question. In this note, I shall dicuss the related series

$$f(z) = \sum_{n \in N} z^n$$

with which σ is connected by the relation

$$\sigma = \int_0^1 \frac{f(z)}{z} dz.$$

I shall prove that if z is an algebraic number such that

$$0 < |z| < 1,$$

then $f(z)$ is a transcendental number; and a similar result holds for infinitely many similar functions.

1. **The problem.** Let $q \geqslant 2$ be a fixed positive integer. Every non-negative integer n can be written in a unique way as a q-adic sum

$$n = h_0 + h_1 q + \ldots + h_r q^r = (h_0, h, \ldots, h_r),$$

where $h_0, h_1 \ldots, h_r$ are integers $0, 1, \ldots, q-1$, and where, in particular, $h_r \neq 0$. For $n = 0$, we write $0 = (0)$. Let

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k be a fixed one of the integers $0, 1, \ldots, q-1$, and let $\mathcal{N}(k)$ be the set of all those integers $n \geqslant 0$ whose digits h_ρ are all different from k,

$n = (h_0, h_1, \ldots, h_r) \geqslant 0,\ 0 \leqslant h_\rho \leqslant q-1,\ h_\rho \neq k\ (\rho = 0, 1, \ldots, r)$.

We shall study here the properties of the generating function

$$f_k(z) = \sum_{n \epsilon N(k)} z^n$$

of $\mathcal{N}(k)$.

2. **The functional equation for** $f_k(z)$. It is clear that $f_k(z)$ is majorized by the series $1+z+z^2+\ldots=(1-z)^{-1}$ and so converges absolutely for $|z| < 1$.

There exists a functional equation between $f_k(z)$ and $f_k(z^q)$ which takes different forms for $k = 0$ and for $k \neq 0$.

I. $k = 0$. If $n = (h_0, h_1, \ldots, h_r)$ belongs to $\mathcal{N}(0)$, then the following two cases arise:

(*i*) $r = 0$, $n = h_0$, so that n is one of the integers $1, 2, \ldots, q-1$.

(*ii*) $r \geqslant 1$, so that n can be written as $n = h_0 + qn'$ where $1 \leqslant h_0 \leqslant q-1$, $n' = (h_1, h_2, \ldots, h_r) \epsilon \mathcal{N}(0)$.

Therefore

$$f_0(z) = \sum_{h_0=1}^{q-1} \left\{ z^{h_0} + \sum_{n' \epsilon N(0)} z^{h_0+qn'} \right\},$$

so that

$$f_0(z) = \frac{z-z^q}{1-z}(1+f_0(z^q)). \qquad \text{(I)}$$

II. $k = 1, 2, \ldots, q-1$. If n belongs to $\mathcal{N}(k)$, then we can write

$$n = (h_0, h_1, \ldots, h_r) = h_0 + qn'$$

where h_0 is one of the integers $0, 1, 2, \ldots, k-1, k+1, \ldots, q-1$, and where

$$n' = (h_1, h_2, \ldots\ h_r) \epsilon \mathcal{N}(k).$$

It is now clear that

$$f_k(z) = \sum_{\substack{h_0=0\\h_0\neq k}}^{q-1} \sum_{n'\varepsilon N(k)} z^{h_0+qn'},$$

whence

$$f_k(z) = \left(\frac{1-z^q}{1-z} - z^k\right) f_k(z^q). \tag{II}$$

The functional equations (I) and (II) may be combined into the one equation

$$f_k(z) = \left(\frac{1-z^q}{1-z} - z^k\right)(\varepsilon_k + f_k(z^q)) \quad (k = 0, 1, \ldots, q-1), \tag{1}$$

where $\varepsilon_k = 1$ if $k = 0$, and $\varepsilon_k = 0$ if $k = 1, 2, \ldots, q-1$. In the simplest case $q = 2$, we have

$$f_0(z) = \sum_{\nu=1}^{\infty} z^{2^\nu-1}, \qquad f_0(z) = z + z f_0(z^2),$$

$$f_1(z) = 1 \qquad , \qquad f_1(z) = f_1(z^2).$$

3. **The analytic behaviour of $f_k(z)$.** It is clear from the definition that

$$f_0(z) = z + z^2 + \ldots + z^{q-1} + \ldots,$$

$$f_k(z) = 1 + z + \ldots + z^{k-1} + z^{k+1} + \ldots \quad (k = 1, \ldots, q-1),$$

whence, for $|z| < 1$,

$$\lim_{\nu\to\infty} f_k(z^{q^\nu}) = 1 - \varepsilon_k \qquad (k = 0, 1, \ldots, q-1). \tag{2}$$

We further deduce from the functional equations (I) and (II) that

$$f_0(z) = \frac{z-z^q}{1-z} + \frac{z-z^q}{1-z}\,\frac{z^q-z^{q^2}}{1-z^q} + \ldots$$
$$+ \frac{z-z^q}{1-z}\,\frac{z^q-z^{q^2}}{1-z^q} \cdots \frac{z^{q^{\nu-1}}-z^{q^\nu}}{1-z^{q^{\nu-1}}}\,(1+f_0(z^{q^\nu})), \tag{3}$$

and

$$f_k(z) = \left(\frac{1-z^q}{1-z} - z^k\right)\left(\frac{1-z^{q^2}}{1-z^q} - z^{kq}\right)\cdots$$
$$\times \left(\frac{1-z^{q^\nu}}{1-z^{q^{\nu-1}}} - z^{kq^{\nu-1}}\right) f_k(z^{q^\nu}), \; (k = 1, 2, \ldots, q-1). \tag{4}$$

THEOREM 1. *If the special case $q=2$, $k=1$ is excluded, then $f_k(z)$ is regular inside the unit circle and has this circle as its natural boundary.*

PROOF. Let κ and λ be two non-negative integers; put

$$\theta = e^{\frac{2\pi i\kappa}{q^\lambda}}.$$

Assume that κ is prime to q so that θ is a primitive q^λ-th root of unity. It is obvious that for $\lambda \geqslant 1$ none of the polynomials

$$\frac{z^{q^{\nu-1}}-z^{q^\nu}}{1-z^{q^{\nu-1}}}, \quad \frac{1-z^{q^\nu}}{1-z^{q^{\nu-1}}}-z^{kq^{\nu-1}} \qquad (\nu = 1, 2, \ldots, \lambda)$$

in z vanishes if $z=\theta$. On the other hand, if the case $q=2$, $k=1$ is excluded, then evidently

$$\lim_{r\to 1} f_k(r) = +\infty \tag{5}$$

as r tends to 1 along the real interval $0 \leqslant r < 1$. But then, by $\theta^{q^\lambda}=1$, from (3), (4), and (5), also

$$\lim_{r\to 1} f_k(\theta r) = \infty.$$

Now the points θ are everywhere dense on the unit circle, and the assertion follows at once.

COROLLARY. *Except for the case $q=2$, $k=1$, $f_k(z)$ is a transcendental function of z.*

4. The arithmetic behaviour of $f_k(z)$. Some twenty years ago, I proved a result in which the following theorem is contained as a special case [*Mathematische Annalen*, 101 (1929), 332-366].

THEOREM 2. *Let $q \geqslant 2$ be a fixed integer, and let*

$$F(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$$

be a power series with the following properties:

(i) *All a_ν are rational numbers.*

(ii) *$F(z)$ converges in a neighbourhood of $z = 0$.*

(iii) *$F(z)$ is not an algebraic function of z.*

(iv) *$F(z)$ satisfies a functional equation of the form*

$$F(z^q) = \frac{a(z)F(z)+b(z)}{c(z)F(z)+d(z)},$$

where $a(z)$, $b(z)$, $c(z)$, $d(z)$ are polynomials with rational coefficients such that $\Delta(z) = a(z)\,d(z)-b(z)\,c(z)$ does not vanish identically in z. Then if z is an algebraic number satisfying

$$0 < |z| < 1, \quad \Delta(z^{q^\nu}) \neq 0 \quad (\nu = 0, 1, 2, \ldots),$$

$F(z)$ is a transcendental number, but not a Liouville number.

If we apply this theorem to $F(z) = f_k(z)$, then

$$a(z) = 1,\ b(z) = -\frac{z-z^q}{1-z},\ c(z) = 0,\ d(z) = \frac{z-z^q}{1-z},$$

or

$$a(z) = 1,\ b(z) = c(z) = 0,\ d(z) = \frac{1-z^q}{1-z}-z^k,$$

according as to whether $k = 0$ or $1 \leqslant k \leqslant q-1$. We therefore obtain the following result.

THEOREM 3. *Let the case $q = 2$, $k = 1$ be excluded. If z is an algebraic number which satisfies the inequality*

$$0 < |z| < 1 \qquad \textit{for } k = 0,$$

and the inequalities

$$0 < |z| < 1,\ \frac{1-z^{q^\nu}}{1-z^{q^{\nu-1}}}-z^{kq^{\nu-1}} \neq 0 (\nu = 1, 2, \ldots) \textit{ for } 1 \leqslant k \leqslant q-1,$$

then $f_k(z)$ is a transcendental number, but not a Liouville number. Furthermore

$$f_k(0) = 1-\varepsilon_k \qquad (k = 0, 1, \ldots, q-1),$$

and if $k = 1, 2, \ldots, q-1$, $0 < |z| < 1$ and there is a $\nu = 1, 2, \ldots$, such that

$$\frac{1-z^{q^\nu}}{1-z^{q^{\nu-1}}}-z^{kq^{\nu-1}}=0,$$

then $f_k(z)=0$.

5. **The zeros of $f_k(z)$.** The polynomials

$$\phi_k(z)=\frac{1-z^q}{1-z}-z^k \qquad (k=1,2,\ldots,q-1)$$

satisfy the functional equations

$$\phi_k(1/z)=z^{-(q-1)}\phi_{q-k-1}(z) \tag{6}$$

Let us assume that $\phi_k(z)$ has $\mu(k)$ zeros of absolute value less than 1, and $\nu(k)$ zeros of absolute value equal to 1. From

$$\phi_{q-1}(z)=1+z+z^2+\ldots+z^{q-2} \qquad (q \text{ arbitrary}),$$
$$\phi_{(q-1)/2}(z)=(1+z+\ldots+z^{(q-3)/2})(1+z^{(q+1)/2}) \qquad (q \text{ odd}),$$

it is clear that

$$\mu(k)=0 \text{ if } k=q-1, \text{ or if } k=(q-1)/2.$$

Further from (6),

$$\nu(k)=\nu(q-k-1). \tag{7}$$

THEOREM 4. *Let* $1\leqslant k\leqslant q-2$ *and* $k\neq(q-1)/2$. *Then* $\mu(k)>0$.

PROOF. The polynomial $\phi_k(z)$ is of exact degree $q-1$; it suffices therefore to prove that $\nu(k)<q-1$. For the product of the zeros of $\phi_k(z)$ is evidently equal to ∓ 1: hence if at least one zero is of absolute value different from 1, then there is also at least one zero of absolute value less than 1.

Since $k\neq(q-1)/2$, it suffices to prove this inequality for $\nu(k)$ if

$$k=1,2,\ldots,[(q-2)/2].$$

We first note that $\phi_k(z)$ has no multiple zeros on the unit circle. For at such zeros,

$$1-z^q-z^k+z^{k+1}=0,\ qz^{q-1}+kz^{k-1}-(k+1)z^k=0,$$

therefore

$$(q-k)z^q=z^{k+1}-k,$$

whence, by $|z|=1$,

$$q-k\leqslant k+1,\quad k\geqslant (q-1)/2,$$

contrary to hypothesis.

Denote by

$$\zeta=e^{\alpha i},\text{ where } 0<\alpha<2\pi,$$

a zero, hence a simple zero, of

$$\phi_k(z)=1+z+\ldots+z^{q-1}-z^k$$

on the unit circle. Since

$$z^{-\frac{q-1}{2}}\phi_k(z)=\frac{z^{\frac{q}{2}}-z^{-\frac{q}{2}}}{z^{\frac{1}{2}}-z^{-\frac{1}{2}}}-z^{-\frac{q-2k-1}{2}},$$

necessarily

$$\frac{\sin q\alpha/2}{\sin\alpha/2}=\cos\frac{q-2k-1}{2}\alpha-i\sin\frac{q-2k-1}{2}\alpha,$$

and so

$$\sin\frac{q-2k-1}{2}\alpha=0.$$

Hence

$$\alpha=\frac{2n\pi}{q-2k-1},$$

where n is one of the integers $1, 2, \ldots, q-2k-1<q-1$. From this the assertion $\nu(k)<q-1$ follows at once.

Let us combine the last results. We have found:

THEOREM 5. *If $k = q-1$, or $k = (q-1)/2$, then $f_k(z)$ has no zeros inside the unit circle. If $k = 0$, then $f_0(z)$ has the algebraic zero $z = 0$, and all its possible other zeros are transcendental. In all other cases, the zeros of $f_k(z)$ are algebraic n mbers, and there are an infinity of them inside the unit circle.*

In a similar way, the generating function of integers with more than one missing digit, or with a missing sequence of digits can be investigated.

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LATTICE POINTS IN A TETRAHEDRON AND GENERALIZED DEDEKIND SUMS

By L. J. MORDELL

Let p, q be two positive integers prime to each other. One form of the reciprocity formula for the so-called Dedekind sums is given by the

THEOREM

$$\sum_{x=1}^{p}\frac{x}{p}\left(\left(\frac{qx}{p}\right)\right)+\sum_{x=1}^{q}\frac{x}{q}\left(\left(\frac{px}{q}\right)\right)=\frac{1}{12}\left(\frac{p}{q}+\frac{q}{p}+\frac{1}{pq}-3\right). \quad (1)$$

Here $((X)) = X-[X]-\frac{1}{2}$, X not an integer,
$= 0$ $\qquad X$ an integer.

Various proofs have been given by Rademacher, Rademacher and Whiteman, Re'dei and myself. For references see[1]. I have shown that the theorem is the particular case $f(x) = x$ in the evaluation of

$$\sum f\left(\frac{x}{p}+\frac{y}{q}\right), \quad (2)$$

where f is a polynomial and the summation is extended over those integer sets (x, y), i.e. lattice points, lying in the triangle

$$0 < x < p,\ 0 < y < q,\ \frac{x}{p}+\frac{y}{q} < 1. \quad (3)$$

This method suggests the extension of the formula (1) to a set of n positive integers $p, q, r, s, \ldots$ no two of which have a common factor. The results, however, now take a different form. Take $n = 3$ and write

$$S_3(p, q, r) = \sum_{x=1}^{p-1}\frac{x}{p}\left(\left(\frac{qrx}{p}\right)\right)+\sum_{x=1}^{q-1}\frac{x}{q}\left(\left(\frac{rpx}{q}\right)\right)$$
$$+\sum_{x=1}^{r-1}\frac{x}{r}\left(\left(\frac{pqx}{r}\right)\right) \quad (4)$$

Denote by $N_3(p, q, r)$ the number of lattice points in the tetrahedron

$$0 \leqslant x < p,\ 0 \leqslant y < q,\ 0 \leqslant z < r,\ 0 < \frac{x}{p}+\frac{y}{q}+\frac{z}{r} < 1. \quad (5)$$

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Write for shortness S_3, N_3. Then we have the

THEOREM.

$$S_3+N_3 = \tfrac{1}{6}pqr+\tfrac{1}{4}\sum qr+\tfrac{1}{4}\sum p+\frac{1}{12}\sum\frac{qr}{p}+\frac{1}{12\,pqr}-2, \qquad (6)$$

the summation referring to p, q, r.

A formula will also be found for $n=4$.

1. The method of proof shows that for $n>4$, the formula for S_n will depend upon the number of lattice points in sections of an n-dimensional tetrahedron defined by

$$\lambda < \Sigma\, x/p < \lambda+1$$

for a number of values of $\lambda \leqslant n-1$.

The function $((X))$ has some well-known properties. It is an odd periodic function of X, and so

$$((-X)) = -((X)),\ ((X+1)) = ((X)).$$

Also

$$((X))+((X+1/p))+\ldots+((X+(p-1)/p)) = ((pX)). \qquad (7)$$

2. We require a formula for N_3. One is given by

$$2N_3 = \sum_{x,\,y,\,z}{}' \left(\left[\frac{x}{p}+\frac{y}{q}+\frac{z}{r}\right]-1\right)\left(\left[\frac{x}{p}+\frac{y}{q}+\frac{z}{r}\right]-2\right), \qquad (8)$$

where the summation is taken over

$$0 \leqslant x < p,\ 0 \leqslant y < q,\ 0 \leqslant z < r, \qquad (9)$$

and the accent denotes the omission of the term $x=y=z=0$.

For from (9), $0 < \frac{x}{p}+\frac{y}{q}+\frac{z}{r} < 3$, and those lattice points for which

$$0 < \frac{x}{p}+\frac{y}{q}+\frac{z}{r} < 1,$$

contribute each 2 to the sum, and each of those for which

$$1 < \frac{x}{p}+\frac{y}{q}+\frac{z}{r} < 3$$

contribute zero. There are no lattice points with

$$\frac{x}{p}+\frac{y}{q}+\frac{z}{r} = 1 \text{ or } 2.$$

We write (8) as

$$2N_3 = \sum_{x,y,z}{}' (E-3/2-((E)))\,(E-5/2-((E))), \qquad (10)$$

where $E = \frac{x}{p}+\frac{y}{q}+\frac{z}{r}$. Hence

$$2N_3 = A+B+C,$$

say, where

$$A = \sum_{x,y,z}{}' (E-3/2)(E-5/2), \quad B = -2\sum_{x,y,z}{}' (E-2)((E)),$$
$$C = \sum_{x,y,z}{}' ((E))^2 \qquad (11)$$

summed over (9).

Hence

$$A+15/4 = \sum_{x,y,z} (E-3/2)(E-5/2).$$

Multiplying out and summing Σx^2, Σxy, Σx, we have, Σ now denoting $\sum_{p,q,r}$,

$$A+\frac{15}{4} = \sum\frac{(p-1)(p)(2p-1)}{6p^2}qr+\sum\frac{2}{4pq}(p)(p-1)(q)(q-1)r$$
$$-4/2\,\Sigma(p)\,(p-1)q/p+15\,pqr/4 = \Sigma\tfrac{1}{3}pqr-\Sigma\tfrac{1}{2}qr+\Sigma\tfrac{1}{6}qr/p$$
$$+\Sigma\tfrac{1}{2}pqr-\Sigma\tfrac{1}{2}(p+q)r+\Sigma\tfrac{1}{2}(r)-2\Sigma pqr+2\Sigma qr+15/4pqr,$$

and so

$$A = \tfrac{1}{4}pqr+\tfrac{1}{2}\Sigma qr+\tfrac{1}{2}\Sigma p+\tfrac{1}{6}\Sigma\, qr/p-15/4. \qquad (12)$$

Clearly we can include $x=y=z=0$ in the summations for B and C in (11).

Next for B. From (7) on summing for y, z in turn, we have

$$\sum_{x,y,z}\frac{x}{p}\left(\left(\frac{x}{p}+\frac{y}{q}+\frac{z}{r}\right)\right) = \sum_{x,y}\frac{x}{p}\left(\left(\frac{rx}{p}+\frac{ry}{q}\right)\right) = \sum_{x}\frac{x}{p}\left(\left(\frac{qrx}{p}\right)\right)$$

since ry runs through a complete set of residues mod q.

Also $$\sum_{x,y,z}\left(\left(\frac{x}{p}+\frac{y}{q}+\frac{z}{r}\right)\right) = \sum_{x=0}^{pqr-1}\left(\left(\frac{x}{pqr}\right)\right),$$

since $qrx+rpy+pqz$ runs through a complete set of residues mod pqr. The sum is

$$\sum_{x=1}^{pqr-1}\left(\frac{x}{pqr}-\tfrac{1}{2}\right)=0.$$

Hence $B=-2\,S_3$. (13)

Finally

$$C=\sum_{x=0}^{pqr-1}\left(\left(\frac{x}{pqr}\right)\right)^2=\sum_{x=1}^{pqr-1}\left(\frac{x}{pqr}-\tfrac{1}{2}\right)^2$$

$$=\frac{1}{6p^2q^2r^2}(pqr-1)(pqr)(2pqr-1)-\frac{1}{2pqr}(pqr-1)(pqr)+\tfrac{1}{4}(pqr-1)$$

$$=\frac{(pqr-1)(2pqr-1)}{6pqr}-\frac{(pqr-1)}{2}+\tfrac{1}{4}(pqr-1)$$

$$=\frac{1}{12}pqr-\tfrac{1}{4}+\frac{1}{6pqr}.$$

Hence we have the required formula

$$2N_3=\frac{1}{3}pqr+\tfrac{1}{2}\Sigma\, qr+\tfrac{1}{2}\Sigma\, p+\frac{1}{6}\Sigma\frac{qr}{p}+\frac{1}{6pqr}-4-2\,S_3,$$

3. For the four dimensional result, let N_4 denote the number of lattice points in

$$0\leqslant x<p,\ 0\leqslant y<q,\ 0\leqslant z<r,\ 0\leqslant w<s,$$

$$0<\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}<1,$$

where no two of p, q, r, s have a common factor. Write

$$S_4=\sum_{p,q,r,s,}\sum_{x=1}^{p-1}\frac{x}{p}\left(\left(\frac{qrsx}{p}\right)\right).$$

We consider now

$$S=\sum_{x,y,z,w}{}'\left(\left[\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}\right]-1\right)\left(\left[\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}\right]-2\right),$$

where the summation is taken over the lattice points L given by

$$0\leqslant x<p,\ 0\leqslant y<q,\ 0\leqslant z<r,\ 0\leqslant w<s$$

with the exclusion of $x=y=z=w=0$.

Here $$0<\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}<4.$$

The points L with
$$1<\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}<3$$
contribute zero to S, while each of the points L with
$$0<\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}<1$$
say N_4 in number, and each of those with
$$3<\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}<4$$
say, N' in number, contribute 2 to S. Hence
$$S=2\ (N_4+N').$$
Let N'' be the number of the points L satisfying
$$0<\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}<1,\ xyzw=0.$$
Then excluding these, we have a 1-1 correspondence between the remaining ones in N_4 and those in N' given by
$$x+x'=p,\ y+y'=q,\ z+z'=r,\ w+w'=s.$$

Hence $N_4=N'+N''$ and $S=4N_4-2N''$.

Now $$N''=N_4''+N_3''+N_2''+N_1'',$$
where N''_r denotes the number of the lattice points L when exactly r of the variables equal zero. Clearly
$$N_3''=\sum_{p,\,q,\,r,\,s}(p-1),\ N_4''=0,$$
also
$$N_2''=\sum_{p,\,q,\,r,\,s}(p-1)\ (q-1)/2,$$
since there are $(p-1)(q-1)$ solutions of
$$0<x<p,\ 0<y<q,\ x/p+y/q<2$$
and there is a 1-1 correspondence between those in $x/p+y/q<1$ and those in $x/p+y/q>1$ given by $x+x'=p, y+y'=q$. Finally $N_1''=\sum_{p,\,q,\,r,\,s} N_3(p,\,q,\,r)$, and so $S=4N_4$
$$-2\sum_{p,\,q,\,r\,\,s}(p-1)-\sum_{p,\,q,\,r,\,s}(p-1)\,(q-1)-2\sum_{p,\,q,\,r,\,s}N_3(p,q,r).\quad(14)$$

Next we split S into three sums, say A', B', C', corresponding to A, B, C but now a fourth variable w/s also occurs. We find $A'+15/4$

$$=\sum\frac{(p-1)(p)(2p-1)qrs}{6p^2}+\sum\frac{2}{4pq}(p)(p-1)(q)(q-1)rs$$

$$-\frac{4}{2}\sum\frac{(p)(p-1)qrs}{p}+\frac{15}{4}pqrs=\sum\frac{(p-1)(2p-1)qrs}{6p}$$

$$+\Sigma\tfrac{1}{2}(p-1)(q-1)rs-2\Sigma(p-1)qrs$$

$$+15\,pqrs/4=pqrs/12+\Sigma\tfrac{1}{2}rs+\Sigma qrs/(6p).$$

Next $$B'=-2S_4.$$

Finally

$$C'=\sum_{x=1}^{pqrs-1}(x-\tfrac{1}{2})^2=\frac{1}{12}pqrs-\tfrac{1}{4}+\frac{1}{6pqrs}.$$

This gives

$$S=\frac{1}{6}pqrs+\tfrac{1}{2}\Sigma pq-4+\sum\frac{qrs}{p}+\frac{1}{6pqrs}-2S_4.$$

Hence on substituting for N_3 from (6) in (14),

$$4N_4+2S_4+2\sum_{p,q,r,s}S_3(p,q,r)=2\Sigma(p-1)+\Sigma(p-1)(q-1)$$

$$+\frac{1}{3}\Sigma pqr+\tfrac{1}{2}\Sigma(qr+rp+pq)+\tfrac{1}{2}\Sigma(p+q+r)$$

$$+\frac{1}{6}\sum\left(\frac{qr}{p}+\frac{rp}{q}+\frac{pq}{r}\right)+\frac{1}{6}\sum\frac{1}{pqr}-16$$

$$+\frac{1}{6}pqrs+\tfrac{1}{2}\Sigma pq+\frac{1}{6}\sum\frac{qrs}{p}-4+\frac{1}{6pqrs}$$

$$=\frac{1}{6}pqrs+2\tfrac{1}{2}\Sigma pq+\tfrac{1}{2}\Sigma p-22+\frac{1}{3}\Sigma pqr$$

$$+\frac{1}{6}\sum\left(\frac{qr}{p}+\frac{rp}{q}+\frac{pq}{r}\right)+\frac{1}{6}\sum\frac{qrs}{p}+\frac{1}{6}\sum\frac{1}{pqr}+\frac{1}{6pqrs}.$$

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A NOTE ON FERMAT'S CONJECTURE

By PAUL TURAN

1. The classical conjecture of Fermat, unpoved so far, asserts that the equation

$$x^q + y^q = z^q \tag{1.1}$$

is unsolvable in positive integers if q is a fixed integer $\geqslant 3$. The history of the (few) attempts, which mean a a progress however slight in this direction, is well known. In this paper, which does not claim to be in this category, we try to modify the problem in a rather plausible way and to give by some quite elementary considerations the first steps in this trend which can be certainly superseded by applying more powerful devices.

2. In what follows we restrict ourselves to the case when q denotes an odd prime; the general case can be reduced easily to it. Denoting by $R_q(N)$ the number of solutions of (1.1) with

$$1 \leqslant x \leqslant N, \quad 1 \leqslant y \leqslant N, \quad 1 \leqslant z \leqslant N, \quad (x, y) = 1, \tag{2.1}$$

the proof of Fermat's conjecture would result:

$$R_q(N) = 0 \qquad \text{for all } N \geqslant 1.$$

Now the question arises: what can be proved actually by way of an upper estimate for $R_q(N)$? I shall show the following

THEOREM. *If $c_1, c_2, \ldots$ denote quantities depending only upon q, we have*

$$R_q(N) < c_1 N \log^{1+\frac{q}{(q-1}} N. \tag{2.2}$$

Replacing the second half of the argument by a well-known but somewhat lengthy one, (2.2) could be improved to

$$R_q(N) < c_2 N \log^{\frac{q}{q-1}} N. \tag{2.3}$$

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3. Now we turn to the proof of the theorem. According to a classical theorem of Euler the prime divisors p of the quantity

$$\frac{x^q + y^q}{x+y}, \qquad (x, y) = 1,\ x \text{ and } y \text{ positive integers},$$

are either q itself or p's with

$$p \equiv 1 \bmod 2q; \tag{3.1}$$

or, as we shall express it, all the prime divisors have the property A. Let the value of z be fixed, say, as

$$z = n,$$

and we write the equation (1.1) in the form

$$(x+y)\left(\frac{x^q + y^q}{x+y}\right) = n^q. \tag{3.2}$$

Owing to (3.1), to any representation of the form (3.2) belongs a representation

$$n^q = k_1\ k_2, \tag{3.3}$$

where k_1, k_2 are integers and all prime factors of k_2 have the property A. [If no prime factors of n have the property A, k_2 means 1]. Conversely to any representation (3.3) we have the system

$$x + y = k_1, \ \frac{x^q + y^q}{x+y} = k_2, \tag{3.4}$$

i.e. to a given pair (k_1, k_2) belong at most $(q-1)$ systems (x, y). Hence denoting by $g(m)$ the number of divisors of m all of whose prime divisors have the property A, we have

$$R_q(N) \leqslant (q-1) \sum_{n=1}^{N} g(n^q). \tag{3.5}$$

[If no prime factors of m have the property A, we define $g(m) = 1$].

4. Let

$$m = q^{\alpha}\, p_1^{\alpha_1} \dots p_r^{\alpha_r}\, q_1^{\beta_1} \dots q_s^{\beta_s},$$

where the p_ν's are all different prime factors of m with

$$p_1 \equiv p_2 \equiv \dots \equiv p_r \equiv 1 \bmod 2q \qquad (4.1)$$

and the q_j's are different from q, from each other and from the p_ν's. Then, we have, obviously

$$g(m) = (1+\alpha)\,(1+\alpha_1)\dots(1+\alpha_r). \qquad (4.2)$$

We form the generating function $F(s)$ of the quantities $g(n^q)$:

$$F(s) = \sum_{n=1}^{\infty} g(n^q) n^{-s},$$

with $s = \sigma + it$, $\sigma > 1$. Since $g(m)$ is multiplicative in the sense

$$g(m_1\, m_2) = g(m_1) g(m_2), \qquad (m_1, m_2) = 1,$$

then owing to the absolute-convergence of the occurring products we have

$$F(s) = \left(1+\frac{q+1}{q^s}+\frac{2q+1}{q^{2s}}+\dots\right) \prod_{p \equiv 1 \bmod 2q}' \left(1+\frac{q+1}{p^s}+\frac{2q+1}{p^{2s}}+\dots\right)$$

$$\times \prod_{p \text{ not} \equiv 1 \bmod 2q}' \left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\dots\right).$$

Hence, choosing

$$\sigma = 1 + \frac{1}{\log N} = \sigma_0,$$

we obtain, owing to the positivity of the $g(n^q)$'s,

$$\frac{1}{eN} \sum_{n \leqslant N} g(n^q) \leqslant \sum_{n \leqslant N} g(n^q) n^{-\sigma_0} < F(\sigma_0) <$$

$$< \left(1+\frac{q+1}{q}+\frac{2q+1}{q^2}+\dots\right) \exp\left(\sum_{p \equiv 1 \bmod 2q} \frac{q+1}{p^{\sigma_0}} + \sum_p \frac{2q+1}{p^2} + \dots\right).$$

$$\exp\left(\sum_{p \text{ not} \equiv 1 \bmod 2q} \frac{1}{p^{\sigma_0}} + \sum_p \frac{1}{p^2} + \sum_p \frac{1}{p^3} + \dots\right)$$

$$< c_3 \exp\left(q \sum_{p \equiv 1 \bmod 2q} p^{-\sigma_0} + \sum_p p^{-\sigma_0}\right)$$

$$< c_4\, \zeta(\sigma_0) \exp\left(q \sum_{p \equiv 1 \bmod 2q} p^{-\sigma_0}\right) < \frac{c_5}{\sigma_0 - 1} \exp\left(q \sum_{p \equiv 1 \bmod 2q} p^{-\sigma_0}\right). \qquad (4.3)$$

But owing to the prime-number theorem we have

$$\Theta(x, 2q, 1) = \sum_{\substack{p \equiv 1 \bmod 2q \\ p \leqslant x}} \log p < \frac{1}{\varphi(2q)} x + c_6 \frac{x}{\log^2(x+1)} = \frac{x}{q-1} + \frac{c_6 x}{\log^2(x+1)};$$

i.e. for $1 < \sigma \leqslant 2$,

$$h(\sigma) = \sum_{p \equiv 1 \bmod 2q} \log p \, . \, p^{-\sigma}$$

$$= \sigma \int_1^\infty \Theta(x, 2q, 1) x^{-\sigma-1} dx < \frac{1}{q-1} \frac{\sigma}{\sigma-1} + c_7 \int_1^\infty \frac{x^{-\sigma} dx}{\log^2(x+1)}$$

$$< \frac{1}{q-1} \cdot \frac{1}{\sigma-1} + 1 + \frac{c_7}{\log^2 2} \cdot \frac{1}{\sqrt{(\sigma-1)}}$$

$$+ \frac{c_7}{\log^2\left(1 + \frac{1}{\sqrt{(\sigma-1)}}\right)} \int_{1/\sqrt{(\sigma-1)}}^\infty x^{-\sigma} dx < \frac{1}{q-1} \cdot \frac{1}{\sigma-1}$$

$$+ \frac{c_8}{(\sigma-1) \log^2 \frac{1}{\sigma-1}}. \tag{4.4}$$

For $\sigma \geqslant 2$ we have trivially

$$h(\sigma) < \sum_{\nu=1}^\infty \log(2\nu q + 1)(2\nu q)^{-\sigma} < c_q (2q)^{-\sigma} \log q. \tag{4.5}$$

From (4.4) and (4.5) we have simply

$$\sum_{p \equiv 1 \bmod 2q} p^{-\sigma_0} = \int_{\sigma_0}^\infty h(\sigma) d\sigma < c_{10} + \int_{\sigma_0}^2 h(\sigma) d\sigma < c_{11} + \frac{1}{q-1} \cdot \log \frac{1}{\sigma_0 - 1}.$$

i.e. from (4.3),

$$\frac{1}{e\mathcal{N}} \sum_{n \leqslant N} g(n^q) < c_{12} (\sigma_0 - 1)^{-1 - \frac{q}{q-1}} = c_{12} \log^{1 + \frac{q}{q-1}} \mathcal{N}.$$

This together with (3.5) proves our theorem.

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ON A PROBLEM IN ELEMENTARY NUMBER THEORY

By T. VIJAYARAGHAVAN

Let A be any positive integer and let $\phi(x, A)$ denote the number of positive integers which are prime to A and less than or equal to x. The object of this note is to prove a result about the upper and lower bounds of $e_1, e_2, e_3, \dots$, where

$$e_n = e_{A,n} = \phi(n, A) - n\frac{\phi(A)}{A}, \quad n = 1, 2, 3, \dots,$$

and $\phi(A)$ stands for $\phi(A, A)$. Erdos [1] raised the question of the order of magnitude of e_n in a paper of his. It is plain that if $p_1, p_2, \dots, p_k$ are all the distinct prime divisors of A, $[x]$ denotes the largest integer not exceeding x and $\{x\}$ denotes $x - [x]$ then

$$\phi(n, A) = n - \sum_r \left[\frac{n}{p_r}\right] + \sum_{r,s}\left[\frac{n}{p_r p_s}\right] \cdots + (-1)^k \left[\frac{n}{p_1 p_2 \dots p_k}\right], \quad \text{(I)}$$

and hence

$$e_n = \sum_r \left\{\frac{n}{p_r}\right\} - \sum_{r,s}\left\{\frac{n}{p_r p_s}\right\} \cdots + (-1)^{k+1} \left\{\frac{n}{p_1 p_2 \cdots p_k}\right\}. \quad \text{(II)}$$

It is easy to see that $e_n = e_{n+p_1 p_2 \cdots p_k} = e_{n+A}$, $n = 1, 2, 3, \dots$,

$$1 - 2^{k-1} \leqslant e_n \leqslant 2^{k-1}, \quad n = 1, 2, 3, \dots.$$

Let $l(A)$ and $m(A)$ denote respectively the maximum and minimum of $e_1, e_2, e_3, \dots$. The following is proved in this note.*

* I am informed by a referee that Prof. D. H. Lehmer has proved by a different method that it is not true that

$$|\phi(x, n) - x\,\phi(n)/n| = o(2^{\nu(n)})$$

and obtained an estimate of

$$\phi(x, n) - x\frac{\phi(n)}{n}.$$

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THEOREM. *If K be any positive integer and δ an arbitrarily chosen positive constant then there corresponds to them an integer A such that*

(i) $\nu(A) = K$, *where* $\nu(A)$ *denotes the number of distinct prime divisors of* A,

(ii) $\dfrac{\phi(A)}{A} > 1-\delta$,

(iii) $l(A) < 1-2^{k-1}+\delta$,

(iv) $m(A) > 2^{k-1}-\delta$.

It may be remarked that *(iii)* is implied by *(iv)* and that therefore it is sufficient to prove *(i)*, *(ii)* and *(iv)*; for from the definition of $\phi(n, A)$ it can be deduced that if $1 \leqslant n \leqslant A$ then $\phi(n, A)+\phi(A-n, A)$ is equal either to $\phi(A)$ or $1+\phi(A)$ according as n is not or is prime to A. Then since

$$\phi(n, A) = e_n+n\frac{\phi(A)}{A}, \ \phi(A-n, A) = e_{A-n}+(A-n)\frac{\phi(A)}{A},$$

it follows that $e_n+e_{A-n} = 0$ or 1 and therefore

$$e_{A-n} = -e_n \text{ or } 1-e_n, \ e_{A-n} \leqslant 1-e_n.$$

Hence if it is proved that $m(A) > 2^{k-1}-\delta$ then we have that for some suitable value of n,

$$1 \leqslant n \leqslant A, \ e_n = m(A) > 2^{k-1}-\delta$$

and hence

$$l(A) \leqslant e_{A-n} \leqslant 1-e_n < 1-2^{k-1}-\delta,$$

i.e. *(iv)* implies *(iii)*.

To prove the theorem for $k = 1$ it is sufficient to observe that if p_1 is prime, $p_1 > 1/\delta$, and $A = p_1$ then

$$\nu(A) = 1, \ e_n = \left\{ n/p_1 \right\}, \ m = e_{p_1-1} = 1-1/p_1 > 1-\delta;$$

also

$$\frac{\phi(A)}{A} = 1-\frac{1}{p_1} > 1-\delta, \ l(A) = e_{p_1} = 0,$$

We complete the proof of the theorem by induction on k; to prove the theorem for k we make use of the inductive hypothesis that to every positive constant δ there corresponds an integer $B = B(k, \delta)$ such that

$$\nu(B) = k-1,\ \phi(B)/B > 1-\delta/6,\ m(B) > 2^{k-2}-\delta/6,$$
$$l(B) < 1-2^{k-2}+\delta/6.$$

It then follows that there exist positive integers a, b such that

$$a < B,\ b < B,\ e_a = e_{B,a} > 2^{k-2}-\delta/6,\ e_b < 1-2^{k-2}+\delta/6.$$

Let p be any prime greater than $B+6B/\delta$ and $n = a+mB$, where m is such that

$$a+mB < (b+1)p \leqslant a+(m+1)B;$$

it is easy to verify that

$$\left[\frac{n}{p}\right] = \left[\frac{a+mB}{p}\right] = b,\quad \left\{\frac{n}{p}\right\} = \left\{\frac{a+mB}{p}\right\} \geqslant 1-\frac{B}{p} > 1-\frac{\delta}{6}.$$

Now since

$$\left[\frac{n}{pt}\right] = \left[\left[\frac{n}{p}\right]\frac{1}{t}\right]$$

it is not difficult to deduce from (I) that

$$\phi(n, Bp) = \phi(n, B) - \phi([n/p], B). \qquad \text{(III)}$$

[This result is also evident from the following considerations. Let $a_1, a_2, \ldots, a_u$ be all those positive integers which are less than or equal to n and prime to B, and let $\alpha_1, \alpha_2, \ldots, \alpha_v$ be the corresponding set of positive integers which are prime to Bp and do not exceed n. The latter set is a subset of the former and is obtained from the former by deleting multiples of p from it; it is to be observed that if a_r is a multiple of p, say, $a_r = xp$ then x is a member of the set $a_1, a_2, \ldots, a_u$.]

And since $n \equiv a \pmod{B}$, $[n/p] = b$ it follows that

$$\phi(n, B) = e_n + n\frac{\phi(B)}{B} = e_a + n\frac{\phi(B)}{B},$$

$$\phi\left(\left[\frac{n}{p}\right], B\right) = e_b + \left[\frac{n}{p}\right]\frac{\phi(B)}{B} = e_b + \frac{n}{p}\frac{\phi(B)}{B} - \left\{\frac{n}{p}\right\}\frac{\phi(B)}{B}.$$

It follows from (III) and the results above that

$$\phi(n, Bp) = e_a - e_b + \left\{\frac{n}{p}\right\}\frac{\phi(B)}{B} + \frac{\phi(B)}{B}\left(n - \frac{n}{p}\right)$$

$$= e_a - e_b + \left\{\frac{n}{p}\right\}\frac{\phi(B)}{B} + n\frac{\phi(Bp)}{Bp},$$

$$e_{Bp,n} = \phi(n, Bp) - n\frac{\phi(Bp)}{Bp} = e_a - e_b + \left\{\frac{n}{p}\right\}\frac{\phi(B)}{B}$$

$$> 2^{k-2} - \frac{\delta}{6} - \left(1 - 2^{k-2} + \frac{\delta}{6}\right) + \left(1 - \frac{\delta}{6}\right)\left(1 - \frac{\delta}{6}\right) > 2^{k-1} - \delta,$$

since

$$e_a > 2^{k-2} - \frac{\delta}{6},\ e_b < 1 - 2^{k-2} + \frac{\delta}{6},\ \frac{\phi(B)}{B} > 1 - \frac{\delta}{6},\ \left\{\frac{n}{p}\right\} > 1 - \frac{\delta}{6}.$$

Writing A for Bp we see that we have proved that $e_{A,n} > 2^{k-1} - \delta$; but plainly $m(A) \geqslant e_{A,n}$ and hence $m(A) > 2^{k-1}\delta$. As already observed it will follow that $l(A) \geqslant 1 - m(A) > 1 - 2^{k-1} - \delta$. Also,

$$\nu(A) = k,\ \frac{\phi(A)}{A} = \frac{\phi(B)}{B}\left(1 - \frac{1}{p}\right) > \left(1 - \frac{\delta}{6}\right)\left(1 - \frac{\delta}{6}\right) > 1 - \delta.$$

The theorem is now evident. I give below

Another proof. It follows easily from the following lemma which is not without a little independent interest.

LEMMA. *Let* $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ *be n subintervals, of the unit interval and* ϵ *the maximum of their lengths; let* $c_1, c_2, \ldots, c_n$ *be n positive integers which are such that*

$$\min\left(c_1, \frac{c_2}{c_1}, \frac{c_3}{c_2}, \ldots, \frac{c_n}{c_{n-1}}\right) > \frac{2}{\epsilon};$$

then there exists an integer x such that

$$a_r \leqslant \{ x/c_r \} \leqslant b_r, \quad r = 1, 2, \ldots, n.$$

PROOF. We say that an interval (a, b) is congruent (mod x) to (c, d) if $\frac{c-a}{x}$, $\frac{d-b}{x}$ are equal integers. Consider now the interval (x_n, y_n), where $x_n = a_n c_n, y_n = b_n c_n$; the length of the interval is plainly equal to

$$c_n(b_n - a_n) \geqslant c_n \epsilon > 2c_{n-1},$$

since $c_n/c_{n-1} > 2/\epsilon$. As (x_n, y_n) has a length greater than $2c_{n-1}$ it contains at least one interval (x_{n-1}, y_{n-1}) which is congruent (mod c_{n-1}) to the interval $(a_{n-1}c_{n-1}, b_{n-1}c_{n-1})$. Similarly we see that the length of (x_{n-1}, y_{n-1}) exceeds $2c_{n-2}$ and that therefore it contains at least one interval (x_{n-2}, y_{n-2}) which is congruent (mod c_{n-2}) to the interval $(a_{n-2}c_{n-2}, b_{n-2}c_{n-2})$, and so on. Thus we obtain a sequence of intervals

$$(x_n, y_n), (x_{n-1}, y_{n-1}), \ldots, (x_2, y_2), (x_1, y_1)$$

which are such that each one but the last contains the succeeding interval; moreover (x_r, y_r) is congruent (mod c_r) to the interval $(a_r c_r, b_r c_r)$, $r = 1, 2, \ldots, n$; and as the length of (x_1, y_1) exceeds 2 it follows that (x_1, y_1) contains at least two integers. If x be one such integer then since x belongs to each one of the intervals (x_r, y_r) we get

$$a_r \leqslant \left\{ \frac{x}{c_r} \right\} \leqslant b_r, \quad r = 1, 2, \ldots, n.$$

This completes the proof of the lemma.

To deduce the theorem from the lemma we shall suppose that $\epsilon = \delta/2^k$, $n = 2^k - 1$,

$$\text{Min}\left(p_1, \frac{p_2}{p_1}, \frac{p_3}{p_1\, p_2}, \ldots, \frac{p_k}{p_1\, p_2 \ldots p_{k-1}}\right) > \frac{2}{\epsilon},$$

$c_1 = p_1$, $c_2 = p_2$, $c_3 = p_1\, p_2$, $c_4 = p_3$, $c_5 = p_1\, p_3$, $c_6 = p_2\, p_3$, $c_7 = p_1\, p_2\, p_3, \ldots$, $c_{n-2} = p_1\, p_3 \ldots p_k$, $c_{n-1} = p_2\, p_3 \ldots p_k$.

The choice $(a_r, b_r) = (1-\epsilon, 1)$ or $(0, \epsilon)$ according as c_r is the product of an odd or even number of prime factors, yields, on account of (II), the result that

$$e_x = e_{p_1 p_2 \ldots p_k, x} > 2^{k-1} - \delta.$$

The theorem now follows easily.

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RAMANUJAN'S TRIGONOMETRICAL SUM AND RELATIVE PARTITIONS

By M. V. Subba Rao

1. This paper arose out of a study of Ramanujan's trigonometrical sum $C_m(n)$ defined by[4]

$$C_m(n) = \sum_{\lambda} \rho^{\lambda n}, \tag{1}$$

where $\rho = e^{2\pi i/m}$ and λ runs though a complete system of representatives of the prime residue classes mod m. The identity of this sum with Von Sterneck's function[1] $f(n, m)$, which is defined as the excess of the number of partitions of n into an even number of parts mod m over those into an odd number of parts, the parts being distinct and chosen from the set of non-zero residues less than m, was first shown by K. G. Ramanathan[3]. It must be noted that in Von Sterneck's function, as also throughout this paper, a partition into zero parts is considered as a partition into an even number of parts. However this case arises only when m divides n.

We give in this paper a direct and general method of dealing with problems of relative partitions. The equivalence of Ramanujan's sum with Von Sterneck's function is a direct consequence of the general Theorem A §3. The method enables us to obtain results analogous to those of Von Sterneck by properly restricting the set of integers from which the elements of the representations are chosen. Finally we apply the method (§5) to obtain the total number of parts in all the partitions of an integer.

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Some of the results of this paper were presented at the Conference of the Indian Mathematical Society at Annamalainagar in 1943. See *Math. Student*, 9 (1943), p. 80.

2. Throughout this paper ρ stands for $e^{2\pi i/m}$, $\sum_{\lambda}$ and $\prod_{\lambda}$ stand for sum and product respectively over a complete system of non-negative residues mod m; $\sum_{\substack{\lambda \\ P}}$, $\prod_{\substack{\lambda \\ P}}$ the sum and product respectively over a complete system of non-negative residues with the property P.

We shall prove certain lemmas which will be used in the sequel.

LEMMA 1.

$$\sum_{\lambda} \rho^{\lambda n} = \begin{cases} m \text{ if } m|n \\ 0 \text{ otherwise.} \end{cases} \tag{2}$$

LEMMA 2. *Let $t|N$. The N numbers $0, 1, \ldots, N-1$ fall into t sets each containing N/t numbers equal to each other* mod t. *Also the $\varphi(N)$ numbers less than and prime to N fall into $\varphi(t)$ sets each set containing $\varphi(N)/\varphi(t)$ numbers equal to each other* mod t.

This lemma is due to Vaidyanathaswamy[5].

LEMMA 3. *If*

$$f_\lambda(m) = \prod_{t} (1+\lambda e^{\frac{2\pi i t}{m}}) \text{ and } \varphi_\lambda(m) = \prod_{\substack{t \\ (t,\, m)=1}} (1+\lambda e^{\frac{2\pi i t}{m}})$$

then

$$\varphi_\lambda(m) = \prod_{d|m} (f_\lambda(d))^{\mu(m/d)}, \tag{3}$$

where μ is the Mobius μ function.

This is the well-known formula for the cyclotomic polynomial.

COR. 1.
$$\varphi_{-1}(m) = e^{\Lambda(m)}, \tag{4}$$
where $\Lambda(m)$ is the well known arithmetic function $= \log p$ *if $m = p^\nu$ and zero otherwise.*

This result is due to Kronecker. See[2].

COR. 2.

$$\varphi_1(m) = \exp\left[-\Lambda(m) + \Lambda\left(\frac{m}{(2, m)}\right)\varphi(m)/\varphi\left(\frac{m}{(2, m)}\right)\right] \tag{5}$$

it being understood that for $m = 1$, $\varphi_1(1) = 2$.

This follows at once from the identity

$$1+x=(1-x^2)/(1-x).$$

LEMMA 4. *If* $G(x)=p_0+p_1x+p_2x^2+\ldots$ *is a polynomial then for* $0\leqslant n<m$

$$p_n+p_{n+m}+p_{n+2m}+\cdots=1/m\sum_k G(\rho^k)\,\rho^{-kn}. \tag{6}$$

This is obtained easily by expanding the right hand-side and using Lemma 1.

3. We shall now prove the main theorems.

THEOREM *A. Let* $\lambda\neq 0$ *be a real number and* $0<a_1<a_2<\ldots<a_r<m$ *be* r *integers. Let* $0\leqslant n<m$ *and* λ^k *the weight attached to a relative partition of* n (mod m) *into* k *parts. Let*

$$P_\lambda(x)=\prod_{l=1}^{r}(1+\lambda x^{a_l})^{\nu}\qquad \nu\geqslant 1 \text{ integral} \tag{7}$$

and $f_\lambda(n, m)$ *the weighted sum of all the partitions of* n (mod m) *into parts chosen from the set* $a_1, \ldots, a_r$, *none of them repeated* ν *times at most. Then*

$$f_\lambda(n, m)=1/m\sum_k P_\lambda(\rho^k)\rho^{-kn}. \tag{8}$$

PROOF. Let $P_\lambda(x)=p_0+p_1x+p_2x^2+\ldots$ Then $P_\lambda(x)$ is a finite polynomial and p_t can be interpreted as the weighted sum of the absolute partitions of t into parts chosen in the way specified in the theorem. Also it is easy to see that

$$f_\lambda(n, m)=p_n+p_{n+m}+p_{n+2m}+\cdots$$

Using Lemma 4 we obtain (8).

THEOREM *B. Let S be the set of integers* $0<a_1<a_2<\ldots<m$ *and* S' *the complement* $0<b_1<b_2<\ldots<b_s<m$ *of S in the set* $1, 2, 3, \ldots, m-1$ *so that* $r+s=m-1$. *Let*

$$Q_\lambda(x)\doteq\prod_{l=1}^{s}(1+\lambda x^{b_l})^{\nu}.$$

Then

$$D_\lambda(m, \delta) = P_\lambda(\rho^k)\, Q_\lambda(\rho^k) = (1-(-\lambda)^{m/\delta})^{\nu\delta}\,(1+\lambda)^{-\nu} \tag{9}$$
where $\delta = (k, m)$.

PROOF. Obviously $P_\lambda(\rho^k) \cdot Q_\lambda(\rho^k) = \prod_{l=1}^{m-1} (1+\lambda\rho^{lk})^\nu$. Using the fact $\prod_l (1+\lambda\rho^l) = 1-(-\lambda)^m$ we obtain

$$\prod_{l=0}^{m-1} (1+\lambda\rho^{lk})^\nu = \prod_{t=0}^{m/\delta} (1+\lambda\mu^t)^{\nu\delta} \qquad \mu = \rho^\delta. \tag{10}$$

It is easily seen that the right side of (10) equals $(1-(-\lambda)^{m/\delta})^{\nu\delta}$. Hence the result.

It needs to be noticed that $P_\lambda(\rho^k) \cdot Q_\lambda(\rho^k)$ depends on k only through its *g. c. d.* (greatest common divisor) with m so that if $P_\lambda(\rho^k)$ depends on k only though its *g. c. d.* with m then so does $Q_\lambda(\rho^k)$. Furthermore in this case using (8) and the definition (1) of Ramanujan's sum it may be easily deduced that if $P_\lambda(\rho^k) = g_\lambda(\delta)$ then

$$f_\lambda(n, m) = \frac{1}{m}\sum P_\lambda(\rho^k)\rho^{-kn} = \frac{1}{m}\sum_{\delta \mid m} g_\lambda(\delta)\, C_{m/\delta}(n) \tag{11}$$

$$\varphi_\lambda(n, m) = \frac{1}{n}\sum_\lambda Q_\lambda(\rho^k)\,\rho^{-kn} = \frac{1}{m}\sum_{\delta \mid m} \frac{D_\lambda(m, m/\delta)}{g_\lambda(m/\delta)} C_\delta(n). \tag{12}$$

4. We deduce now very simple consequences of these theorems by specializing the sets S and S'.

(*a*). Let S be the set of positive integers $1, 2, \ldots m-1$. Then S' is empty. Defining an empty product $Q_\lambda(\rho^k) = 1$ we obtain

$$f_\lambda(n, m) = \frac{(1+\lambda)^{-\nu}}{m} \sum_{\delta \mid m} (-(-\lambda)^{m/\delta})^{\nu\delta}\, C_{m/\delta}(n). \tag{13}$$

Putting $\lambda = -1$ we have

$$D_{-1}(m, \delta) = \begin{cases} 0, & \delta > 1 \\ m^\nu, & \delta = 1 \end{cases} \tag{14}$$

so that

$$f_{-1}(n, m) = \sum (-1)^t = m^{\nu-1}\, C_m(n) \tag{15}$$

where the summation is carried through all representations

of n (mod m) as a sum of integers $1, 2, \ldots, m-1$ each repeated at most ν times.

Von Sterneck's result [1, 4] coresponds to $\nu = 1$ so that the parts in the representations are distinct.

Putting $\lambda = 1$ we have

$$D_1(m, \delta) = \begin{cases} 0, & m/\delta \text{ even} \\ 2^{\nu(\delta-1)}, & m/\delta \text{ odd}. \end{cases} \tag{16}$$

Hence

$$f_1(n, m) = \Sigma\, 1 = \frac{1}{2^\nu m} \sum_{\substack{\delta \mid m \\ m/\delta \text{ odd}}} 2^{\nu\delta} C_{m/\delta}(n) \tag{17}$$

the summation being extended over all representations of n as sum of integers $1, 2, \ldots, m-1$ each integer repeated at most ν times.

The result (17) for $\nu = 1$ is proved in [1, 4].

(*b*). Let S now be the set of integers less than and prime to m. Then

$$P_\lambda(\rho^k) = \prod_{\substack{t \\ (t, m)=1}} (1 + \lambda\, \rho^{kt})^\nu = \sum_{\substack{t \\ (t, m/\delta)=1}} \left(1 + \lambda e^{\frac{2\pi i t}{m/\delta}}\right)^{\frac{\nu\varphi(m)}{\varphi(m/\delta)}}. \tag{18}$$

This is just a consequence of Lemma 2. Using now Lemma 3 with m/δ in place of m we obtain

$$P_\lambda(\rho^k) = \prod_{d \mid m/\delta} \left(1 - (-\lambda)^d\right)^{\nu\mu(m/d\delta)\varphi(m)/\varphi(m/\delta)}. \tag{19}$$

Let now $\lambda = -1$. Then by (4) we see that

$$P_{-1}(\rho^k) = \exp\left[\nu\Lambda(m/\delta)\varphi(m)/\varphi(m/\delta)\right]. \tag{20}$$

Thus

$$f_{-1}(n, m) = \Sigma\, (-1)^t = 1/m \sum_{\delta \mid m} C_\delta(n) \exp\left[\frac{\nu\Lambda(\delta)\varphi(m)}{\varphi(\delta)}\right] \tag{21}$$

and the summation in the middle term extends over all representations of n as a sum (mod m) of integers less than and prime to m each of these integers repeated at most ν times.

For $\nu = 1$ this result is proved in[4]

Let now $\lambda = 1$. Let us interpret $\Delta(m/t) = 0$ if t does not divide m then it is easy to see that in (5) we may write $\varphi_1(m) = e^{\Lambda(m/2)}$. Thus

$$P_1(\rho^k) = \exp\left[\nu\Lambda(m/2\delta)\varphi(m)/\varphi(m/\delta)\right] \tag{22}$$

so that

$$f_1(n, m) = \Sigma\, 1 = 1/m \sum_{\delta \mid m} C_\delta(n) \exp\left[\nu\Lambda(\delta/2)\,\varphi(m)/\varphi(\delta)\right], \tag{23}$$

where we have the interpretation that if $\delta = 1$ then the first term is $2^{\nu\varphi(m)}$. The summation in the middle term is over all representations of $n(\mathrm{mod}\ m)$ in terms of integers less than and prime to m and repetitions being not allowed more than ν times.

If we follow the methods in [2] it is easy to see that the result (5) gives the formula

$$C_m(1) - \tfrac{1}{2}C_m(2) + \tfrac{1}{3}C_m(3) - \ldots = \Lambda\left(\frac{m}{2}\right),\ m > 2. \tag{24}$$

This is analogous to a result by Ramanujan [5].

(c). Let us now try to compute $Q_\lambda(\rho^k)$ which is not a constant function as in (a). Obviously from definition

$$Q_\lambda(\rho^k) = \prod_{\substack{t \\ 1<(t,\, m)<m}} \left(1 + \lambda e^{\frac{2\pi i t k}{m}}\right)^\nu \tag{25}$$

$$= \prod_{\substack{d \mid m \\ 1<d<m}} \ \prod_{\substack{t \\ (t,\, m/(m,\, d\delta))=1}} \left(1 + \lambda e^{\frac{2\pi i t}{r}}\right)^{\nu_\delta \frac{\varphi(m/d)}{\phi(m/(d\delta,\, m))}}$$

$r = \dfrac{m}{(d\delta, m)}$. This is obtained by using Lemma 2.

Let now $\lambda = -1$. Then from (14) it follows

$$Q_{-1}(\rho^k) = \begin{cases} 0, & \delta > 1 \\ m^\nu e^{-\nu\Lambda(m)}, & \delta = 1, \end{cases} \tag{26}$$

so that we have

$$\phi_{-1}(n, m) = \Sigma\,(-1)^t = m^{\nu-1} C_m(n)\, e^{\pm\nu\Lambda(m)} \tag{27}$$

the summation extended over all representations of n (mod m) into parts which are less than and *not* prime to m and which are repeated at most ν times.

Let now $\lambda = 1$. Then from (24) and (16) we obtain the result

$$Q_1(\rho^k) = \begin{cases} 0 & , m/\delta \text{ even} \\ 2^{\nu(\delta-1)} & , m/\delta \text{ odd and} > 1 \\ 2^{\nu(m-\phi(m)-1)}, & \delta = m. \end{cases} \qquad (28)$$

We therefore obtain

$$\phi_1(n, m) = \sum 1 = \frac{1}{2^\nu m} \sum_{\substack{\delta \mid m \\ m \mid \delta \text{odd} > 1}} 2^{\nu\delta} C_{n/\delta}(n) + \left(\frac{2^{\nu(m-\phi(m)-1)} - 1}{m} \right), \qquad (29)$$

where the middle term is a sum over all representations as a sum into parts which are *not* prime to m and which are not repeated more than ν times.

5. In § 3 we had been concerned with weighted partitions where the weight attached to a partition into k parts was $\lambda^k = e^{k \log \lambda}$. Instead we now consider weights k or $(-1)^k k$ which will enable us to get results analogous to those in § 4. Moreover these weights lend themselves to simple arithmetical interpretation. For instance, attaching a weight k to a partition into k parts is the same as counting the number of parts in a partition so that the weighted sum of all the partitions, the weight attached to a partition into k parts being k, and satisfying certain conditions, is the same as the total number of parts in all the partitions satisfying those conditions. To compute this total number of parts we proceed in the following manner.

The functions $f_\lambda(n, m)$ and $\phi_\lambda(n, m)$ of § 3, 4 are polynomials in λ. Regarding λ as a variable and differentiating these functions with regard to λ we see that $\frac{d f_\lambda}{d\lambda}$ can be regarded as a weighted sum of the partitions satisfying the conditions of Theorem A, but the weight now being $k \lambda^{k-1}$ with a partition into k parts. Thus the

total number of parts can be obtained as soon as we put $\lambda = 1$ after differentiation. Thus we get

THEOREM C. *The total number of parts in all the partitions of* n (mod m), $n \neq 0$ *into parts chosen from* $0 < a_1 < a_2 < \ldots < a_r < m$ *each being repeated at most* ν *times is*

$$(d/d\lambda)\,(f_\lambda(n, m)). \tag{30}$$

When $n = 0$ we have to differentiate only that part of $f_\lambda(n, m)$ containing non-zero powers of λ. In other words, the total number of parts in all the partitions of m would be

$$(d/d\lambda)\,(f_\lambda(m, m)) + f_0(m, m). \tag{31}$$

We may remark that by successive differentiation it is possible to compute the values of the weighted sums with weights (k/r), for fixed r, for partitions into k parts, (k/r) being the usual binomial coefficient. But these weighted sums do not lend themselves to simple arithmetical interpretation as the case above with $r = 1$.

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ON HELLY'S THEOREM AND THE AXIOMS OF CONVEXITY

By F. W. LEVI

In 1913, E. Helly[1] has established: "When $C_1, \ldots, C_m$ are convex domains in an n-dimensional Euclidean space and every set of $n+1$ of these domains have a common point, then there exists a point which is common to all the domains." Recently[6] this theorem has obtained an enhanced importance because of the numerous applications which it affords. The continuity of the Euclidean space and its metric properties are essential for the proofs [2, 4, 6, 7] published upto now. In this paper, it will be shown that the range of validity of Helly's theorem is much larger. It includes, e.g. all the spaces which satisfy the axioms of incidence and of order. Furthermore the theorem holds in its generalised form (Theorem H) in arbitrary sets in which a family of subsets which satisfies two axioms, is distinguished. It appears that Theorem H is one of the basic statements of a geometry in which the "macroscopic" notions of straight lines, planes, vectors, etc. are not used and where convexity is a fundamental notion. In §1 the axioms of convexity will be formulated and discussed. §2 contains the proof of Theorem H. The proof of Helly's theorem for spaces in which the axioms of incidence and order hold, will be given in §3, applications of the theorem especially to group-theorey in §4. Fundamental properties of a geometry based on the notion of of convexity will be sketched in §5.

1. Given a set S of elements, called points, in which a class $\mathscr{C}$ of subsets is distinguished. If $\mathscr{C}$ satisfies $\mathbf{C}_1$ and $\mathbf{C}_2$, the elements of $\mathscr{C}$ will be called *semiconvex* sets. If $\mathscr{C}$ satisfies $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$ and $\mathbf{C}_4$, they will be called *convex* sets. Thus convexity implies semiconvexity.

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C$_1$. If $K_\nu \in \mathfrak{C}$, then $\cap_\nu K_\nu \in \mathfrak{C}$ for every (not necessarily countable) aggregate $\{K_\nu\}$.

From **C$_1$** it follows that when $S' \subseteq S$ is a subset of a set of the class $\mathfrak{C}$, the meet of all the sets of that class which contain S' is a set of the class $\mathfrak{C}$ which is uniquely determined by S'. It will be denoted by $|S'| \supseteq S'$. When S' is a finite set, say $S' = \{P_1, \ldots, P_n\}$, we shall use the abbreviated notation

$$|\{P_1, \ldots, P_n\}| = |P_1 \ldots P_n|. \tag{1}$$

Given n arbitrary points $P_1, \ldots, P_n$, the set (1) may not exist. However any statement about a set of type (1) in this paper implies the assumption of existence. From the definition of (1) it follows that

$$|P_1 \ldots P_n| \subseteq |QP_1 \ldots P_n|,$$

where equality holds if and only if $Q \in |P_1 \ldots P_n|$. In particular $|P_1 \ldots P_n| = |P_1 \ldots P_n P_n|$. Furthermore (1) is invariant for permutation of the P's.

C$_2$. Let $m > n$ and $Q_1, \ldots, Q_m \in |P_1 \ldots P_n|$; then there exists a permutation $i_1, \ldots, i_m$ of $1, \ldots, m$ and a positive integral number $r < m$ such that

$$|Q_{i_1} \ldots Q_{i_r}| \cap |Q_{i_{r+1}} \ldots Q_{i_m}| \neq 0, \tag{2}$$

where 0 denotes the vacuous set.

C$_3$. Let $Q \in |P_1 \ldots P_n| = K$ and for $i = 1, \ldots, n$,

$$K_i = |P_1 \ldots P_{i-1} Q P_{i+1} \ldots P_n|;$$

then

$$K = \cup_i K_i. \tag{3}$$

[From the definition follows $K \supseteq \cup_i K_i$; therefore (3) could be replaced by $K \subseteq \cup_i K_i$.]

C$_4$. Let $R \in |P_1 P_2| \subseteq |P_1 P_2 P_3|$; then

$$|RP_3| = |RP_1P_3| \cap |RP_2P_3|. \tag{4}$$

From the definition follows an inequality only where "$=$" in (4) is replaced by $\subseteq$.

2. We suppose now that S is a set in which a class $\mathfrak{C}$ of *semiconvex* sets is determined.

THEOREM H. *Let* $N > n$ *and* $K_1, \ldots, K_N \subseteq |P_1 \ldots P_n|$ *be semiconvex sets; if for every set* $i_1, \ldots, i_n$ *of different numbers* $1, \ldots, N$, $K_{i_1} \cap \ldots \cap K_{i_n} \neq 0$, *then* $K_1 \cap \ldots \cap K_N \neq 0$.

PROOF. The proposition is obvious for $N = n$. To prove it for $N > n$, we use mathematical induction; therefore it suffices to show that when $k > n$, $D_j = K_1 \cap \ldots \cap K_{j-1} \cap K_{j+1} \cap \ldots \cap K_k \neq 0$, for $j = 1, \ldots, k$, implies $K_1 \cap \ldots \cap K_k \neq 0$. We therefore select $Q_j \in D_j$ for $j = 1, \ldots, k$. Then $Q_j \in K_t$ for $t \neq j$. If now $j_1, \ldots, j_k$ is a permutation of $1, \ldots, k$, then $Q_{j_1}, \ldots, Q_{j_r} \in K_{j_s}$ for $s > r$ and therefore $|Q_{j_1} \ldots Q_{j_r}| \subseteq K_{j_{r+1}} \cap \cdots \cap K_{j_k}$. Similarly $|Q_{j_{r+1}} \ldots Q_{j_k}| \subseteq K_{j_1} \cap \ldots \cap K_{j_r}$. Hence

$$|Q_{j_1} \ldots Q_{j_r}| \cap |Q_{j_{r+1}} \ldots Q_{j_k}| \subseteq K_1 \cap \ldots \cap K_k.$$

As $\mathbf{C}_2$ holds in S, there exists a permutation of the sub-scripts, such that the left side is non-vacuous; therefore the right side is non-vacuous. Hence Theorem *H* holds.

If in the supposition of Theorem *H*, the number N is replaced by ∞, the meet of every finite number of the domains is non-vacuous; however the meet of all of them may be vacuous. If, in particular, S is a topological space, the domains are all closed and a finite sub-family of them has a compact meet, then there exists a common point of all the domains. This can be proved by exactly the same argument as was used by Rademacher and Schoenberg for deriving Theorem 2 of [6] from Theorem 1.

3. Let S be an n-dimensional space in which the axioms of incidence and Hilbert's [3, §3] axioms of order hold. To express that B lies "between" A and C, we write

$$A \circ B \circ C. \tag{5}$$

Obviously (5) is equivalent with $C \circ B \circ A$. A set $T \subseteq S$ is called "convex" if $A, C \in T$ implies $B \in T$, when B satisfies (5). For this "convexity", $\mathbf{C}_1$ obviously holds.

That $\mathbf{C}_3$ and $\mathbf{C}_4$ are satisfied can be shown easily, but is not interesting for the following investigation. It will be shown that $\mathbf{C}_2$ holds and therefore Theorem H is valid for all these spaces. We use the notation (1) for the smallest "convex" set which contains $P_1, \ldots, P_n$; then $|P_1 \ldots P_n|$ is contained in an $(n-1)$-dimensional subspace of S. To prove the validity of $\mathbf{C}_2$ it therefore suffices to show that (2) can be satisfied whenever $Q_1, \ldots, Q_m$ lies in a subspace of dimension $< m-1$. Moreover for $k < m$,

$$|Q_{i_1} \ldots Q_{i_r}| \cap |Q_{i_{r+1}} \ldots Q_{i_k}| \subseteq |Q_{i_1} \ldots Q_{i_r}| \cap |Q_{i_{r+1}} \ldots Q_{i_m}|.$$

Therefore it suffices to verify (2) under the condition that $Q_1 \ldots, Q_m$ span up a space of dimension $m-2$, whereas every subset of $t < m$ of these points spans up a space of $t-1$ dimensions. We consider separately $m=3$, $m=4$ and $m>4$.

(1) $m=3$. The three points are collinear; hence $Q_{i_1} \circ Q_{i_2} \circ Q_{i_3}$ holds. Therefore

$$|Q_{i_1} Q_{i_3}| \cap |Q_{i_2}| = |Q_{i_2}| \neq 0.$$

(2) $m=4$. The points are coplanar. An elementary consideration based on the axioms of order shows that either one point lies inside the triangle formed by the three other points, i.e. $|Q_{i_1} Q_{i_2} Q_{i_3}| \cap |Q_{i_4}| = |Q_{i_4}| \neq 0$, or the points consist of two pairs which bound mutually intersecting segments; therefore

$$|Q_{i_1} Q_{i_2}| \cap |Q_{i_3} Q_{i_4}| \neq 0.$$

(3) $m>4$. The dimension of the space S is $\geqslant 3$. Therefore Desargues' theorem holds in S; if the parallel-axiom holds, S is a space of $d = m-2$ dimensions over a skew-field F. We consider at first the case when F is an ordered field. Then the multiplicative group of the non-zero elements of F has a sub-group P of index 2; its coset N contains -1. Both P and N are semimodules. The elements of P are called *positive*, those of N *negative*, elements. If $a_1 + \ldots + a_r = 0$, then either $a_1 = \ldots = a_r = 0$,

or at least one term is positive and at least one term is negative. When F is an ordered field, the axioms of order can be satisfied in the following way which tallies with the usual definitions of Euclidean geometry and which we shall call the *natural ordering.* Denote the points of S by their coordinates $(x) = (x_1, \ldots, x_d)$, $x_i \epsilon F$; then $(x) \circ (y) \circ (z)$ if and only if $(y) = (\kappa x + \lambda z)$, κ, λ not in N,

$$\kappa + \lambda = 1. \qquad (6)$$

Put $Q_i = (x^i), r \leqslant m$; then $(y) \epsilon | Q_1 \ldots Q_r |$ if and only if $(y) = (\kappa_1 x^1 + \ldots + \kappa_r x^r)$, $\kappa_1, \ldots, \kappa_r$ not in N, $\Sigma \kappa_i = 1$.

As $Q_1, \ldots, Q_m$ are linearly dependent, but no proper subset of them is, there exists a relation

$$\Sigma a_i x_j^i = 0, j = 1, \ldots, d, \ \Sigma a_i = 0, \ a_i \neq 0 \text{ for } i = 1, \ldots, m. \quad (7)$$

As some of the coefficients a are positive and some negative, (7) can be expressed in the form

$$(\kappa_1 x_j^{i_1} + \ldots + \kappa_r x_j^{i_r}) - (\kappa_{r+1} x_j^{i_{r+1}} + \ldots + \kappa_m x_j^{i_m}) = 0, \ j = 1, \ldots, d,$$

$$1 \leqslant r < m, \ \kappa_1, \ldots, \kappa_m \epsilon P, \ \kappa_1 + \ldots + \kappa_r = \kappa_{r+1} + \ldots + \kappa_m = 1.$$

Hence $(z) = (\kappa_1 x^{i_1} + \ldots + \kappa_r x^{i_r}) = (\kappa_{r+1} x^{i_{r+1}} + \ldots + \kappa_m x^{i_m})$

$$\epsilon | Q_{i_1} \cdots Q_{i_r} | \cap | Q_{i_{r+1}} \cdots Q_{i_m} | \neq 0.$$

Thus $\mathbf{C}_2$ holds in naturally ordered spaces, in particular in Euclidean geometry and therefore Helly's theorem holds in these cases. However a between-ness satisfying Hilbert's axioms of order may be defined also in essentially different ways, even when F is an ordered field, e.g. F may admit a non-trivial automorphism. Whether all the changes in the ordering of the space S can be generated by (1, 1)-mappings of the points of S, is not certain; therefore the fact that F does not admit any automorphism different from the identity, is not sufficient for proving the uniqueness of the ordering in S. However we can prove:

Lemma. *The d-dimensional ($d > 1$) geometry over the field of the rational numbers admits only the natural ordering.*

Proof. Let S_0 be a d-dimensional geometry over the rational number-field and between-ness be defined in

S_0 such that the axioms of order are satisfied. S_0 can be extended to a projective geometry S'_0. If $A, B, C \in S_0$ are collinear, and X is the point at infinity of the straight line, then $A \circ B \circ C$ is equivalent with the condition that the pairs of points A, C and B, X separate mutually. By projection and intersection, pairs of mutually separating points are transformed into such pairs. If therefore a sequence of projections and intersections transforms $A, B, C, X \rightarrow A', B', C', X$, then $A \circ B \circ C$ implies $A' \circ B' \circ C'$. Hilbert's addition and multiplication of segments are generated each by a finite number of projections and intersections and therefore they leave between-ness invariant. If we represent the points on the x-axis by their coordinates, then $a \circ b \circ c$ implies $(ra+s) \circ (rb+s) \circ (rc+s)$ for arbitrary rational numbers r, s. Therefore $2 \circ 0 \circ 1$ implies $0 \circ 2 \circ 1$ (for $r = -1$, $s = 2$) and conversely; hence only the order-relation $0 \circ 1 \circ 2$ ($= 2 \circ 1 \circ 0$) is admissible. Therefore $(r-1) \circ r \circ (r+1)$ and so one sees that the points with integral coordinates are ordered according to the natural ordering. Given three rational numbers $a = u:t > b = v:t > c = w:t$ (u, v, w integral, $t > 0$), then $u \circ v \circ w$ and therefore $a \circ b \circ c$ hold. Thus the points on the x-axis are ordered by the natural ordering and by linear transformation it follows that S_0 admits only the natural ordering.

Again we consider a space S of dimension $d > 2$ in which the axioms of incidence and of order are valid, but is arbitrary otherwise. In S, the points $Q_1, \ldots, Q_m$, $m = d+2$ with the properties as supposed before, are given. If the parallel-axiom does not hold in S, we can extend S to a projective space of d dimensions and by omitting one of the ideal $(d-1)$-dimensional hyperplanes, obtain a d-dimensional space S' in which the axioms of incidence and order as well as the parallel axiom hold[5]. S' is a d-dimentional geometry over a skew-field F. The prime field F is denoted by F_0. The points $Q_1, \ldots, Q_m$ generate a d-dimensional geometry S'_0 which is a sub-

geometry of S'. When the axioms of order hold in a geometry, they hold also in every subgeometry; therefore they hold in S_0', hence every segment in S_0' contains an infinite number of points; thus F_0 is an infinite field, i.e. F_0 is isomorphic to the field of the rational numbers. For the convexity in the geometries S, S_0' and $S_0 = S \cap S_0'$ we use the notations $| \ldots |$, $| \ldots |_0'$ and $| \ldots |_0$ respectively. By the adjunction of the ideal elements to S which resulted in the construction of the geometry S', no new points [5, §§6, 7 and 9, in particular p. 60] have been added to the segments of S. If therefore $P_1, \ldots, P_t \in S_0$ then

$$|P_1 \ldots P_t|_0' = |P_1 \ldots P_t|_0 \subseteq |P_1 \ldots P_t|. \qquad (8)$$

From the lemma it follows that $\mathbf{C}_2$ holds in S_0' and therefore there exists a permutation $i_1, \ldots, i_m$ of $1, \ldots, m$ and a positive $r < m$ such that

$$0 \neq |Q_{i_1} \cdots Q_{i_r}|_0' \cap |Q_{i_{r+1}} \cdots Q_{i_m}|_0' \subseteq |Q_{i_1} \cdots Q_{i_r}| \cap |Q_{i_{r+1}} \cdots Q_{i_m}|.$$

Thus $\mathbf{C}_2$ holds in S. So we obtain the result:

In every space in which the axioms of incidence and of order are valid, the axioms $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *are satisfied and therefore Helly's theorem holds.*

4. Helly's theorem has been applied to prove [6] that certain families of sets have common elements. However it can also be used conversely to show that certain families of sets cannot be mapped simultaneously on convex (or semiconvex) sets by any (1, 1) mapping of the elements. Such statements seem to be interesting as no topological properties of the (1, 1) mapping are assumed.

Example. The straight lines of a plane π are mapped (1, 1), but otherwise arbitrarily, on a plain point set T. The lines intersecting a square Σ_ν in π are mapped on Q_ν, where ν runs over all the real numbers. There exist quadruplets of squares which are not intersected by the

same straight lines, whereas every subset of three of them are hit by a common straight line (e.g. four white squares on a chess-board which have a corner in common with a fifth white square). Therefore at least one of the corresponding sets Q_ν is non-convex. One can prove easily that every point of T, with the possible exception of a single one [it is not known to the author whether mappings with such an exceptional point exist] lies in a non-convex Q. Similarly for the lines which intersect circles or ellipses.

Let G be a group; we consider the subsets of G which consist of the subgroups when in each subgroup the unit element is omitted. These sets form a family $\mathfrak{C}$ which satisfies $\mathbf{C}_1$; if and only if $\mathfrak{C}$ satisfies also $\mathbf{C}_2$, the group G will be called *semiconvex*. In a semiconvex group, Theorem H holds.

We prove now: *Every free Abelian group with n free generators is semiconvex.*

Proof. The statement is obvious for $n = 1$. We suppose it to be correct for $n < m$ and prove it for $n = m$. The subgroups $\neq \{1\}$ of a free Abelian group with m free generators are free Abelian groups A_k with $k \leqslant m$ free generators. When we use the notation (1) there is therefore

$$|a_1 \ldots a_t| = A_k - \{1\},\ k \leqslant m,\ k \leqslant t.$$

If $N > t \geqslant k$, $b_1, \ldots, b_N \in (A_k - \{1\})$; then $|b_1 \ldots b_N| = A_u - \{1\}$, where $u \leqslant k \leqslant t < N$. Suppose

$$|b_1 \ldots b_r| \cap |b_{r+1} \ldots b_N| = 0,\ 0 < r < N$$

$$|b_1 \ldots b_r| \cup \{1\} = A_{k_1},\ |b_{r+1} \ldots b_N| \cup \{1\} = A_{k_2}.$$

Then $k_1 \leqslant r$, $k_2 \leqslant N - r$. Moreover

$$A_{k_1} A_{k_2} = A_u,\ A_{k_1} \cap A_{k_2} = \{1\}.$$

Hence $A_u = A_{k_1} \times A_{k_2}$; therefore $k_1 + k_2 = u < N$. Thus one of the inequalities $k_1 < r$, $k_2 < N - r$ holds, say $k_1 < r$.

Then it follows from the supposition that for some positive $v < r$ and a suitable permutation of $1,\dots,r$,

$$|b_{i_1}\dots b_{i_v}| \cap |b_{i_{v+1}}\dots b_{i_r}| \neq 0$$

and therefore

$$|b_{i_1}\dots b_{i_v}| \cap |b_{i_{v+1}}\dots b_{i_r} b_{r+1}\dots b_N| \neq 0.$$

Thus A_m is semiconvex. So we get the result:

If $C_1,\dots,C_N$ are subgroups of a free Abelian group A_n and every $(n+1)$-tuplet of those subgroups has an element $\neq 1$ in common, then $\cap_j C_j \supset \{1\}$.

Another application of Theorem H is the following:

S consists of the real polynomials

$$f(x) = a_n x^n + \dots + a_0,\ a_n > 0.$$

The subsets $K_\nu \subseteq S$ are those which satisfy the condition: "If $f_1(x), \dots, f_m(x) \in K_\nu$, $m = 1, 2, \dots$, then $\sum_1^m \lambda_i f_i(x) \in K_\nu$ when $\lambda_1, \dots, \lambda_m > 0$." The sets K_ν can easily be shown to be semiconvex and therefore Theorem H holds in S.

5. Whereas in the earlier portions of this paper, the axioms $\mathbf{C}_1$ and $\mathbf{C}_2$ only have been used, we shall now consider systems Σ in which a family of convex sets satisfying all the four axioms of convexity is distinguished. It will be shown that when Q is not in $|R|$, the set $|QR|$ admits a linear ordering of the "primitive particles" $|P|$. By the homomorphic mapping $P \to |P|$ one obtains therefore a geometry in which there exist "segments" $|QR|$, but not necessarily "straight lines". The segments may not admit a continuation beyond their endpoints and if so, the continuation may not be unique. Therefore the plain axiom of order ([3], II, 4) has no meaning in this

geometry. Some kind of substitute for it, is the theorem that two transversals of a triangle must meet. This proposition appears below as Theorem 3.

THEOREM 1. $|PR| = |PQ|$ *implies* $|R| = |Q|$.

PROOF. As $Q \epsilon |PR|$, $|QR|$ exists. $|PR| \cap |QR| = |PQ| \cap |QR| = |QPQ| \cap |QRQ| = |QQ| = |Q|$. From the symmetry $Q \longleftrightarrow R$ it follows that $|PQ| \cap |QR| = |R|$. Hence $|Q| = |R|$.

COROLLARY 1. $Q \epsilon |P|$ *implies* $|Q| = |P|$.

PROOF. $|PP| = |P| = |PQ|$; hence $|P| = |Q|$.

COROLLARY 2. *Every convex set is invariant for the mapping*: $\cup_\nu \{P_\nu\} \rightarrow \cup_\nu |P_\nu|$ *of the subsets of* Σ.

THEOREM 2. *Let* $Q_0, \ldots, Q_{m+1} \epsilon |Q_0 Q_{m+1}|$ *and the sets* $|Q_0|, \ldots, |Q_{m+1}|$ *all be different; then* $|Q_0 Q_{m+1}| = |Q_0 Q_{i_1}| \cup |Q_{i_1} Q_{i_2}| \cup \ldots \cup |Q_{i_m} Q_{m+1}|$, *where* $i_1, \ldots, i_m$ *is a permutation of* $1, \ldots, m$, $i_0 = 0$, $i_{m+1} = m+1$ *and*

$$|Q_{i_{k-1}} Q_{i_k}| \cap |Q_{i_k} Q_{i_{k+1}}| = |Q_{i_k}|, \; k = 1, \ldots, m$$
$$|Q_{i_{k-1}} Q_{i_k}| \cap |Q_{i_{k+r}} Q_{i_{k+r+1}}| = 0 \text{ for } r > 0.$$

PROOF. (By mathematical induction).

For $m = 1$, $Q_1 \epsilon |Q_0 Q_2| = |Q_0 Q_1 Q_2| = |Q_0 Q_1 Q_1| \cup |Q_2 Q_1 Q_1|$ (for (3)) $= |Q_0 Q_1| \cup |Q_1 Q_2|$. Moreover for (4):

$$|Q_0 Q_1 Q_1| \cap |Q_2 Q_1 Q_1| = |Q_1 Q_1| = |Q_1|.$$

Suppose the theorem holds for $m < k$.

$$|Q_0 Q_{k+1}| = |Q_0 Q_{i_1}| \cup \ldots \cup |Q_{i_{k-1}} Q_{k+1}|,$$

where $i_1, \ldots, i_{k-1}$ is a suitable permutation of $1, \ldots, k-1$ and the conditions for $\cap$ hold as required. As $|Q_k| \neq |Q_i|$,

for $i \neq k$, Q_k is not a point of any $|Q_i|$ and therefore situated in one and only one $|Q_{i_r}Q_{i_{r+1}}|$. Then

$$|Q_{i_r}Q_{i_{r+1}}| = |Q_{i_r}Q_k| \cup |Q_kQ_{i_{r+1}}|$$

and

$$|Q_k| = |Q_{i_r}Q_k| \cap |Q_kQ_{i_{r+1}}|.$$

Thus the conditions for $\cap$ and $\cup$ hold for $m = k$. Hence the theorem.

THEOREM 3. *Let* $|ABC|$ *exist,* $D \in |AC|$, $E \in |BC|$; *then* $|AE| \cap |BD| \neq 0$.

PROOF. As $A, B, D, E \in |ABC|$, it follows from $\mathbf{C}_2$ that when the theorem is not correct, at least one of the following six alternatives must hold:

(1) $D \in |ABE|$ (2) $B \in |ADE|$ (3) $|AB| \cap |DE| \neq 0$
(4) $|AD| \cap |BE| \neq 0$.

(1′) $E \in |ABD|$ (2′) $A \in |BDE|$.

It will be proved that each of these alternatives implies $|AE| \cap |BD| \neq 0$. For reasons of symmetry, the consideration of (1′) and (2′) is not necessary.

(1) Suppose $D \in |ABE|$. As $D \in |AC| \subseteq |ACE|$, $D \in |ABE| \cap |ACE| = |AE|$. Hence $D \in |AE| \cap |BD| \neq 0$.

(2) Suppose $B \in |ADE| \subseteq |ACE|$; hence $B \in |ABE| \cap |ACE| = |AE|$. Hence $B \in |AE| \cap |BD| \neq 0$.

(3) Suppose $Q \in |AB| \cap |DE|$. As $|AB| \subseteq |BAD|$ and $|DE| \subseteq |BCD|$, $Q \in |BAD| \cap |BCD| = |BD|$; by symmetry: $Q \in |AE|$. Hence $Q \in |AE| \cap |BD| \neq 0$.

(4) Suppose $T \in |AD| \cap |BE|$; then $T \in |BAD| \cap |BCD| = |BD|$; by symmetry: $T \in |AE|$.

Hence $T \in |AE| \cap |BD| \neq 0$; hence Theorem 3.

Addition. At the time of proof-reading, the author became aware of two publications of Richard Rado which are closely connected with this paper: "A theorem on general measure, *J. London Math. Soc.*, 21 (1946), 291-300" and "A theorem on Abelian groups, *J. London Math. Soc.* 22 (1947), 219-226". The starting point of Rado's invertigations is very different from the author's but the results mutually overlap to a certain extent.

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SUR LE THEOREME DE LEBESGUE-NIKODYM (IV)

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1. Nous nous occupons, dans ce travail, de la généralisation du théorème de Lebesgue-Nikodym aux fonctions à valeurs dans un *espace de Banach*. Nous garderons en principe les notations de [2]; E est donc un ensemble quelconque, μ une mesure (positive) définie sur une tribu (σ-corps ou σ-algèbre) $\mathcal{T}$ de parties de E, telle que E soit mesurable et de mesure finie (on généraliserait très aisément tout ce qui suit au cas où E est réunion dénombrable d'ensembles de mesure finie). $L^1(E, \mu)$ (resp. $L^\infty(E, \mu)$), ou simplement L^1 (resp. L^∞) désigne l'espace de Banach des *classes* de fonctions sommables (resp. mesurables et essentiellement bornées) dans E; pour une fonction f dont la classe appartient à L^1 (resp. L^∞), nous désignerons cette classe par $\tilde{f}$. La norme d'une classe dans L^1 (resp. L^∞) se notera $N_1(\tilde{f})$ (resp. $N_\infty(\tilde{f})$).

Soit F un espace de Banach, F' son dual; une application $\mathbf{g}$ de E dans F' est dite *faiblement mesurable* si, pour tout vecteur $\mathbf{a} \in F$, la fonction numérique $\langle \mathbf{a}, \mathbf{g}(t) \rangle$ est mesurable dans E; elle est *fortement bornée* si la fonction numérique $\|\mathbf{g}(t)\|$ est bornée dans E. La généralisation "idéale" du théorème de Lebesgue-Nikodym aux fonctions vectorielles serait la suivante:

THEOREME A. *Toute application linéaire continue de l'espace de Banach L^1 dans l'espace de Banach F' est de la forme*

$$\tilde{f} \to \int f(t)\, \mathbf{g}(t)\, d\mu \qquad (1)$$

où $\mathbf{g}$ est une application faiblement mesurable et fortement bornée de E dans F' (le second membre de (1) étant une intégrale *faible*, c'est-à-dire l'unique élément de F' tel que

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$$\langle \mathbf{x}, \int f(t)\, \mathbf{g}(t) d\mu \rangle = \int f(t) \langle \mathbf{x}, \mathbf{g}(t) \rangle d\mu. \quad (2)$$

Ce théorème est connu lorsque F est un espace de Banach *séparable* (théorème de Dunford-Pettis, voir [2] et [4]). Je ne suis pas parvenu à le démontrer dans tous les cas; mais l'analyse qui va suivre montrera mieux pourquoi le cas de Dunford-Pettis est particulièrement simple; elle mettra d'autre part en lumière le fait que le théorème, sous sa forme générale, repose, non sur des propriétés de l'espace de Banach F, mais bien sur des propriétés de l'espace L^∞ des classes de fonctions *numériques* mesurables et essentiellement bornées (c'est-à-dire en fait des propriétés de la mesure μ).

2. Soit U une application linéaire continue de L^1 dans F', il existe donc un nombre $m \geqslant 0$ tel que l'on ait $\| U(\overleftarrow{f}) \| \leqslant m . N_1(\overleftarrow{f})$, d'où résulte, pour tout $\mathbf{x} \in F$, $| \langle \mathbf{x}, U(\overleftarrow{f}) \rangle | \leqslant m . N_1(\overleftarrow{f}) . \| \mathbf{x} \|$; l'application $\overleftarrow{f} \rightarrow \langle \mathbf{x}, U(\overleftarrow{f}) \rangle$ est donc une forme linéaire *continue* sur l'espace L^1, et il résulte du théorème de Lebesgue-Nikodym classique qu'il existe une *classe* $\overleftarrow{g}_{\mathbf{x}}$ appartenant à L^∞, et telle que, pour toute fonction $g_{\mathbf{x}}$ de cette classe, on ait

$$\langle \mathbf{x}, U(\overleftarrow{f}) \rangle = \int f(t)\, g_{\mathbf{x}}(t) d\mu(t) \quad (3)$$

avec en outre

$$N_\infty(\overleftarrow{g}_{\mathbf{x}}) \leqslant m . \| \mathbf{x} \|. \quad (4)$$

La comparaison des formules (3) et (2) montre que, si le théorème A est vrai, la fonction $g_{\mathbf{x}}^*(t) = \langle \mathbf{x}, \mathbf{g}(t) \rangle$ appartient à la classe $\overleftarrow{g}_{\mathbf{x}}$; en outre, il est clair que l'application $\overleftarrow{g}_{\mathbf{x}} \rightarrow g_{\mathbf{x}}^*$ est *linéaire*. Si H désigne le sous-espace de L^∞, image de F par l'application $\mathbf{x} \rightarrow \overleftarrow{g}_{\mathbf{x}}$, nous voyons donc que si le théorème A est vrai, il existe pour chaque classe $\overleftarrow{g} \in H$ un *représentant* g^* appartenant à cette classe, telle que l'application $\overleftarrow{g} \rightarrow g^*$ soit *linéaire*. Inverse-

ment, supposons qu'il y ait une telle application, *telle en outre que* $|g^*(t)| \leqslant \mathcal{N}_\infty(\overset{\frown}{g})$ *pour tout* $t \in E$; alors, il est clair d'après (4) que, pour tout $t \in E$, $\mathbf{x} \to g_{\mathbf{x}}^*(t)$ est une forme linéaire continue sur F, qu'on notera $\mathbf{g}(t)$ (élément de F') ; on a par suite $<\mathbf{x}, \mathbf{g}(t)> = g_{\mathbf{x}}^*(t)$, donc $\mathbf{g}$ est faiblement mesurable, et la relation (4) montre en outre que $\|\mathbf{g}(t)\| \leqslant m$ pour tout $t \in E$: on obtient donc le théorème A.

Cette analyse montre aussitôt pourquoi la démonstration du th. A est particuliérement simple lorsque F est *séparable*: dans ce cas, H, image continue de F, est un sous-espace *séparable* de L^∞ et la correspondance $\overset{\frown}{g} \to g^*$ est alors facile à définir *dans* H. On prendra une suite partout dense $(\overset{\frown}{g}_n)$ d'éléments linéairement indépendants de H, et pour chaque n une fonction g_n^* de la classe $\overset{\frown}{g}_n$; pour chaque système fini $(r_1, \ldots, r_n)$ de nombres rationnels, on supprime les points de E (formant un ensemble de mesure nulle) tels que $|\sum_i r_i g_i^*(t)| > \mathcal{N}_\infty(\sum_i r_i \overset{\frown}{g}_i)$; en retranchant de E un ensemble de mesure nulle, on peut donc supposer que $|\sum_i r_i g_i^*(t)| \leqslant \mathcal{N}_\infty(\sum_i r_i \overset{\frown}{g}_i)$ pour tous les systèmes (r_i) et tout point $t \in E$; l'application $\sum_i r_i \overset{\frown}{g}_i \to \sum_i r_i g_i^*$ dans l'espace des fonctions bornées sur E est alors uniformément continue sur l'ensemble des $\sum_i r_i \overset{\frown}{g}_i$ et se prolonge par suite par continuité à H en donnant l'application cherchée.

Il est clair qu'on ne peut songer à procéder de la sorte lorsque F n'est plus séparable; mais le th. A sera établi pour un espace F *quelconque* chaque fois que, pour la mesure μ, le théorème suivant sera vrai :

THEOREME B.—*Il existe une application linéaire* $\overset{\frown}{g} \to g^*$ *de l'espace* $L^\infty(E, \mu)$ *dans l'espace des fonctions bornées sur* E, *telle que* g^* *appartienne à la classe* $\overset{\frown}{g}$, *et que* $|g^*(t)| \leqslant \mathcal{N}_\infty(\overset{\frown}{g})$ *pour tout* $t \in E$.

3. Le th. B a été démontré par J. von Neumann [7] lorsque μ est la mesure de Lebesgue sur l'intervalle $I = [0, 1]$. Nous allons rappeler le principe de sa démonstration, que les notions modernes de topologie permettent d'abréger notablement. Pour tout point t de I, soit $V_n(t)$ l'intersection de I et de l'intervalle de milieu t et de longueur $1/n$, et posons

$$g_n(t) = \frac{1}{\mu V_n(t)} \int_{V_n(t)} g d\mu$$

pour toute fonction g mesurable et essentiellement bornée. Il résulte du théorème de recouvrement de Vitali que la suite des fonctions $g_n(t)$ converge *presque partout* vers $g(t)$ [8]; en outre, on a évidemment $|g_n(t)| \leqslant \mathcal{N}_\infty(\tilde{g})$ pour tout $t \in I$, et g_n ne dépend que de la *classe* $\tilde{g}$ de g. Cela étant, soit $\mathcal{F}$ un *ultrafiltre* plus fin que le filtre de Fréchet sur **N** ([1], p. 21-26); comme en tout point $t \in I$, les $g_n(t)$ forment une suite bornée par $\mathcal{N}_\infty(\tilde{g})$, les $g_n(t)$ convergent en tout point $t \in I$, *suivant l'ultrafiltre* $\mathcal{F}$, vers une limite $g^*(t)$ telle que $|g^*(t)| \leqslant \mathcal{N}_\infty(\tilde{g})$ ([1], p. 59). En outre, en tout point t où la *suite* $g_n(t)$ est convergente, $g^*(t)$ est égal à sa limite, donc $g^*(t) = g(t)$ presque partout, en d'autres termes g^* appartient à la classe $\tilde{g}$. Il est clair par ailleurs que l'application $\tilde{g} \to g^*$ est linéaire, et vérifie donc toutes les conditions du th. *B*.

4. On peut aisément, en utilisant des résultats de P. Halmos et J. von Neumann [5], étendre les théorèmes *A* et *B* à des espaces mesurés *E* plus généraux que l'intervalle *I*. De facon précise, supposons que l'espace $L^1(E, \mu)$ soit *séparable*; alors, on peut décomposer E en deux ensembles E_0 et E_1, E_0 étant dénombrable et la mesure induite sur E_1 étant non-atomique (tout point est de mesure nulle); le th. *B* étant trivial pour E_0, on peut donc se réduire au cas où la mesure μ est non-atomique dans E. On peut en outre supposer que $\mu E = 1$.

Alors, l'algèbre booléienne mesurée $B(E)$ des classes d'ensembles mesurables contenus dans E est *isomorphe* à l'algèbre mesurée B (I) des classes d' ensembles mesurables (pour la mesure de Lebesgue) contenus dans I: en d'autres termes, il existe entre ces deux algèbres une correspondance biunivoque, qui conserve les opérations de réunion et intersection dénombrable, et conserve la mesure. En outre, il résulte du travail de Halmos et von Neumann ([5], p. 335-336) qu'il existe une application (non biunivoque) ϕ de E dans I (non nécessairement *sur* I), qui "réalise" l'isomorphie précédente des algèbres booléiennes au sens suivant: pour tout ensemble M mesurable dans I, $\phi^{-1}(M)$ est mesurable dans E, et sa classe correspond dans l'isomorphie à celle de M.

On peut alors identifier de facon évidente les espaces L^{∞} sur E et sur I, et faire correspondre à toute classe $\tilde{g}$ de L^{∞} la fonction $g^*(\phi(t))$ définie dans E; il est clair que cette correspondance satisfera aux conditions du th. B. Ce dernier, et par suite aussi le th. A, sont donc démontrés *lorsque l'espace* $L^1(E, \mu)$ *est séparable.*

5. Nous n'insisterons pas sur les diverses applications du th. de Dunford-Pettis, où, grace à ce qui précède, on peut supprimer l'hypothèse de séparabilité pour l'espace de Banach F, pourvu qu'on sache que $L^1(E, \mu)$ est séparable. Signalons seulement en particulier que (moyennant la meme hypothèse sur E), cela donne le *dual* de l'espace de Banach $\mathscr{L}^1(E, F)$ des fonctions *fortement sommables* définies dans E et à valeurs dans F: toute forme linéaire continue sur cet espace s'écrit

$$\mathrm{f} \to \int < \mathrm{f}(t), \mathrm{g}(t) > d\mu,$$

où g est une fonction faiblement mesurable et essentiellement bornée, à valeurs dans le dual F' de F; la démonstration de ce théorème, donnée dans [2] (th. 3) lorsque F est séparable, est applicable encore sans modification essentielle au cas que nous envisageons ici.

6. Comme autre application importante, nous allons généraliser le théorème établissant l'existence de la décomposition d'une mesure, démontré dans [2] (th. 5) en supposant l'espace *E compact et métrisable* (μ étant une mesure de Radon sur E). Nous allons voir que le théorème subsiste lorsqu'on suppose seulement que l'*espace quotient* E/R est *compact et métrisable*, E étant un espace compact *quelconque*. Nous allons appliquer le th. A en prenant pour F l'espace $\mathcal{C}(E)$ des *fonctions continues numériques* dans E ; son dual F' est donc l'espace $\mathcal{M}(E)$ des *mesures de Radon* sur E. Soient ϕ l'application canonique de E sur E/R, λ la mesure quotient de μ sur E/R ; à toute fonction numérique f sommable sur E/R (pour la mesure λ) correspond la fonction $f \circ \phi$ sur E, sommable pour μ, et telle que

$$\int (f \circ \phi)\, d\mu = \int f\, d\lambda.$$

L'application qui, à toute fonction continue $h \in \mathcal{C}(E)$, fait correspondre l'intégrale $\int (f \circ \phi)\, h\, d\mu$ est une mesure de Radon sur E, car on a

$$\left| \int (f \circ \phi)\, h\, d\mu \right| \leqslant \|h\| . \int |f|\, d\lambda ;$$

désignons-la par $V(\tilde{f})$ (elle ne dépend évidemment que de la classe de f). On a donc

$$\int (f \circ \phi)\, h\, d\mu = \langle h, V(\tilde{f}) \rangle, \text{et } |\langle h, V(\tilde{f}) \rangle| \leqslant \|h\| . \mathcal{N}_1(\tilde{f});$$

ceci ayant lieu pour toute fonction $h \in \mathcal{C}(E)$, on a $\|V(\tilde{f})\| \leqslant \mathcal{N}_1(\tilde{f})$; autrement dit, V est une application linéaire continue de $L^1(E/R)$ dans $F' = \mathcal{M}(E)$. Comme $L^1(E/R)$ est séparable, le th. A montre qu'il existe une application $z \to \nu_z$ de E/R dans $\mathcal{M}(E)$, faiblement mesurable et fortement bornée (par 1) telle que

$$V(\tilde{f}) = \int f(z)\, \nu_z d\lambda(z) ;$$

ce qui signifie encore que pour toute fonction $h \in \mathcal{C}(E)$, on a

$$\int (f \circ \phi) h \, d\mu = \int f(z) \, d\lambda(z) \int h \, d\nu_z. \tag{5}$$

Il est facile de voir en premier lieu que pour tout $z \in E/R$, la mesure ν_z sur E est *positive*, autrement dit qu'on a $\int h \, d\nu_z \geqslant 0$ pour toute fonction continue $h \geqslant 0$ dans E. En effet, la méthode de démonstration du th. A à partir du th. B, dans le cas actuel, consiste à remarquer que l'on a $< h, V(\tilde{f}) > = \int f g_h \, d\lambda$, où $\tilde{g}_h$ est un élément bien déterminé de $L^\infty(E/R, \lambda)$; on a donc

$$\int (f \circ \phi) \, h \, d\mu = \int f g_h \, d\lambda$$

pour toute fonction $f \in L^1(E/R, \lambda)$; $f \geqslant 0$ entraîne par suite $\int f g_h \, d\lambda \geqslant 0$, ce qui prouve que la classe $\tilde{g}_h$ est *positive*. La méthode de démonstration du th. B montre alors que, pour tout $z \in E/R$, $\int h \, d\nu_z$ est limite suivant un ultrafiltre, d'une famille de fonctions *positives* dans E/R ; on a donc $\int h \, d\nu_z \geqslant 0$ pour tout $z \in E/R$, ce qui démontre notre assertion.

Reste enfin à montrer que, pour presque tout $z \in E/R$, la mesure ν_z est *portée par l'ensemble fermé* $\phi^{-1}(z)$. Nous allons montrer d'abord que, si A est un ensemble *fermé* dans E/R, pour presque tout $z \in A$, le support de la mesure ν_z est contenu dans $\phi^{-1}(A)$. Si G est l'ensemble ouvert complémentaire de $\phi^{-1}(A)$ dans E, il suffit pour cela de prouver que $\nu_z(G) = 0$ pour presque tout $z \in A$. Or, on a $G = \phi^{-1}(B)$, où B est l'ensemble ouvert complémentaire de A dans E/R; et comme E/R est métrisable, la fonction caractéristique χ_B de l'ensemble ouvert B est enveloppe supérieure d'une suite croissante de fonctions continues g_n dans E/R. Appliquons la formule (5) en remplaçant f par la fonction caractéristique χ_A, et h par la fonction $g_n \circ \phi$; comme $g_n(\phi(x)) = 0$ dans $\phi^{-1}(A)$, le premier

membre de (5) est nul, d'où $\int_A d\lambda(z) \int (g_n \circ \phi)\, d\nu_z = 0$. Pour tout $z \in A$, $\int (g_n \circ \phi)\, d\nu_z$ tend vers $\nu_z(G)$ lorsque n croît indéfiniment, en vertu du th. de Lebesgue ; le meme théorème prouve que l'on a, à la limite $\int_A \nu_z(G)\, d\lambda(z) = 0$, ce qui montre bien que $\nu_z(G) = 0$ presque partout dans A.

Pour achever notre démonstration, il suffit de remarquer qu'il existe dans l'espace compact métrisable E/R une suite (A_n) d'ensembles fermés telle que tout point de E/R soit intersection d'une suite extraite de (A_n) (il suffit de prendre pour les A_n les adhérences d'une base dénombrable de E/R). Pour chaque indice n, il existe un ensemble de mesure nulle (pour λ) $H_n \subset A_n$ tel que pour tout $z \in A_n$ n'appartenant pas à H_n, le support de ν_z soit contenu dans $\phi^{-1}(A_n)$. Soit H l'ensemble de mesure nulle réunion des H_n ; pour tout z n'appartenant pas à H, le support de ν_z est contenu dans tous les $\phi^{-1}(A_n)$ tels que $z \in A_n$; comme l'intersection de ces ensembles est $\phi^{-1}(z)$, le support de ν_z est bien contenu dans $\phi^{-1}(z)$.

7. On est naturellement amené à se demander si le th. B (et par suite le th. A) est vrai pour *tout* espace mesuré E (de mesure finie). La méthode du n°4 permet de ramener ce problème au cas d'espaces E particuliers. En effet, D. Maharam a démontré[6] que tout espace mesuré E, de mesure finie, est réunion d'une famille dénombrable (E_n) d'ensembles sans point commun, tels que E_0 soit dénombrable, et que pour chaque E_n, l'algèbre booléienne mesurée $B(E_n)$ soit isomorphe à l'algèbre booléienne des classes d'ensembles mesurables dans un *produit* I^M d'intervalles I, où chaque facteur est muni de la mesure de Lebesgue, et où M est un ensemble de puissance quelconque (après avoir éventuellement multiplié la mesure dans E_n par un facteur convenable). En reprenant le raisonnement de D. Maharam, il n'est pas difficile en outre de former une application ϕ de E_n dans

I^M telle que, pour tout ensemble mesurable A dans I^M, $\phi^{-1}(A)$ soit mesurable dans E_n et que sa classe corresponde dans l'isomorphie à celle de M. Cela étant, il est clair qu'il suffit de démontrer le th. B pour chacun des espaces E_n, et le raisonnement du *n*° 4 montre que cela revient à le démontrer *pour tout espace* I^M.

Pour un tel espace, on saurait démontrer le th. B si on pouvait démontrer l'existence, en chaque point $t \varepsilon I^M$, d'un ensemble filtrant décroissant $\mathcal{B}(t)$ d'ensembles mesurables de mesure $>$ 0 ayant la propriété suivante: pour toute fonction mesurable et bornée g dans E, et tout ensemble $V(t) \varepsilon \mathcal{B}(t)$, posons

$$s_{V(t)} = \frac{1}{\mu V(t)} \int_{V(t)} g\, d\mu ;$$

alors, pour presque tout t, la limite de $s_{V(t)}$ suivant l'ensemble filtrant $\mathcal{B}(t)$, existe et est égale à $g(t)$. Comme pour tout $t \varepsilon E$, il est clair que $s_{V(t)}$ ne dépend que de la classe $\tilde{g}$ de g, et que $|s_{V(t)}| \leqslant \mathcal{N}_\infty(\tilde{g})$, on définirait une application $\tilde{g} \rightarrow g^*$ ayant les propriétés requises, en procédant comme au *n*° 3 : pour chaque $t \varepsilon E$, on considérerait un ultrafiltre $\mathcal{J}(t)$ plus fin que le filtre des sections de $\mathcal{B}(t)$, et on prendrait pour $g^*(t)$ la limite de $s_{V(t)}$ suivant cet ultrafiltre.

Toute la difficulté consiste donc à définir les " systèmes dérivants " $\mathcal{B}(t)$. Une conjecture naturelle est la suivante : pour toute partie finie J de M, soit J′ son complémentaire dans M ; on peut écrire $I^M = I^J \times I^{J'}$; pour tout $t \varepsilon E$, toute partie finie J de M et tout entier n, désignons par $V_{n,J}(t)$ le produit de $I^{J'}$ et du cube de côté $1/n$ dont le centre est la projection de t sur I^J; l'ensemble $\mathcal{B}(t)$ des $V_{n,J}(t)$ est filtrant décroissant, et on pourrait penser que ces ensembles filtrants possèdent la propriété voulue. Or, j'ai démontré dans un autre travail [3] que cette conjecture est *inexacte*. Le problème de l'existence des $\mathcal{B}(t)$ reste donc entier lorsque M n'est pas dénombrable.

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A DIOPHANTINE PROPERTY OF SOME SUMMABLE FUNCTIONS

By J. F. Koksma

1. Introduction. Let $g(t)$ denote a function of the class L^2 in $0 \leqslant t \leqslant 1$ with period 1. Let $\{ f_n(x) \}$ $(n = 1, 2, \ldots)$ denote a sequence of real functions of x $(a \leqslant x \leqslant b)$. In a former paper [5] I proved that if $f_n(x)$ satisfies some general conditions, the relation

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} g(f_n(x)) = \int_0^1 g(t)dt \tag{1}$$

holds for almost all values of x in $a \leqslant x \leqslant b$, assuming that the coefficients c_k in the Fourier expansion

$$g(t) \approx \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikt} \tag{2}$$

have the property

$$\sum_{p=1}^{\infty} \frac{R_p}{p} < \infty, \text{ where } R_p = \sum_{k=p+1}^{\infty} |c_k|^2. \tag{3}$$

The above meant conditions are satisfied, e.g. by the functions $f_n(x) = nx$; $f_n(x) = \lambda_n x$ (where $\lambda_1 < \lambda_2 < \cdots$ denotes any sequence of real numbers with $\lambda_{n+1} - \lambda_n \geqslant c > 0$); $f_n(x) = x^n$ $(x \geqslant 1)$, etc.

In this paper I treat the very special case $f_n(x) = nx$. Using an entirely different method (i.e. the method meant in [5], p. 962 and p. 357 respectively) I prove the following

Theorem 1. *Let $g(t) \in L^2$ denote a periodic function with period* 1, *such that the coefficients c_k in its Fourier expansion* (2) *satisfy the relation*

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$$\sum_{k=1}^{\infty} |c_k|^2 \omega(k) < \infty \tag{4}$$

where

$$\omega(k) = \sum_{d \mid k} \frac{\log^2 d}{d}.$$

Then almost everywhere,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(n\,x) = \int_0^1 g(t)\,dt. \tag{5}$$

Remarks: I. It is clear that, without loss of generality, we may restrict ourselves to *real* functions $g(t)$, such that

$$c_0 = \int_0^1 g(t)\,dt = 0.$$

Hence

$$g(t) \approx \sum_{k=-\infty}^{\infty} c_k\, e^{2\pi i k t} \quad (c_0 = 0,\ c_{-k} = \bar{c}_k). \tag{6}$$

II. One can easily prove that $\omega(k) = O((\log\log k)^4)$. From this the reader easily deduces that one can replace the condition (4) by the condition that the series

$$\sum_{(p)} R_p \frac{(\log\log p)^3}{p \log p} \tag{7}$$

converges, assuming that $R_p (\log\log p)^4 \to 0$.

Therefore Theorem 1 is stronger than the corresponding special case $f_n(x) = nx$ of the general theorem mentioned above.

The condition (7), and therefore also the condition (4), obviously will be fulfilled if we assume that for a constant $\epsilon > 0$,

$$\int_0^1 |g(t) - S_p(t)|^2\,dt = 2\,R_p = O\left(\frac{1}{(\log\log p)^{4+\epsilon}}\right).$$

Erdös [1] under the weaker condition

$$R_p = O\left(\frac{1}{(\log\log p)^{1+\varepsilon}}\right),$$

proved that (1) is true for almost all x for any function of the type $f_n(x) = \lambda_n\,x$, assuming that $\lambda_1, \lambda_2, \ldots$ is a lacunary sequence of increasing integers, i.e. $\lambda_{n+1}/\lambda_n > 1+\delta$ for

some positive constant δ. It is clear, that Theorem 1 is not included in this result, as the sequence 1,2,3,...is not lacunary.

[As to the history of the above problem, I remark that the lacunary case $f_n(x) = \lambda_n x$ ($\lambda_{n+1}/\lambda_n \geqslant 1+\delta$) had already been treated by Kac, Salem and Zygmund [3] in 1948. In the special case $f(x) = a^n x$ (a denoting a fixed integer $\geqslant 2$) Raikoff [6] had even proved in 1936 that (1) holds for any periodic function which is L-integrable. Riess [7] pointed out that Raikoff's theorem is actually an instance of the ergodic theorem and from a theorem of Erdös [1] it is clear that one may not expect such a nice result for the general case $f_n(x) = \lambda_n x$. The special case $f_n(x) = nx$ had been considered already in 1922 by Khintchine [4] who assumed that $g(x)$ in $0 \leqslant x \leqslant 1$ denoted the characteristic function of a given measurable set E in $0 \leqslant x \leqslant 1$.]

2. Some lemmas.

LEMMA 1. *Let $g(x) \in L^2$ denote the function of Theorem 1, having the Fourier series* (6), *then for $M \geqslant 0$, $N \geqslant 1$*

$$\int_0^1 \Big| \sum_{n=M+1}^{M+N} g(nx) \Big|^2 dx = 2 R_0 N + 2\,\Omega(M, N),$$

where

$$\Omega(M, N) = \sum_{d=1}^{\infty} \sum_{\sigma=1}^{N-1} \sum_{\substack{(u,v)=1 \\ M+1 \leqslant \sigma u \leqslant M+N \\ M+1 \leqslant \sigma v \leqslant M+N}} c_{ud}\, c_{vd} \tag{8}$$

(the inner sum is to be extended over all couples $u \neq v$ satisfying the inequalities $M+1 \leqslant \sigma u \leqslant M+N$, $M+1 \leqslant \sigma v \leqslant M+N$, and being relatively prime).

PROOF. For $n = 1, 2, \ldots$ the functions

$$g_n(x) = g(nx) \approx \sum_{k=-\infty}^{\infty} \gamma_k\, e^{2\pi i k x}$$

belong to the class L^2, whereas

$$\gamma_k = \int_0^1 g(nx)\, e^{-2\pi ikx}\, dx = \begin{cases} c_{k/n} & \text{if } k \equiv 0 \pmod n \\ 0 & \text{if } k \not\equiv 0 \pmod n \end{cases}$$

Hence ($M \geqslant 0$, $N \geqslant 1$ denoting integers)

$$\sum_{n=M+1}^{M+N} g(nx) \approx \sum_{k=-\infty}^{\infty} A_k\, e^{2\pi ikx},$$

where

$$A_0 = 0,\ A_k = \sum_{\substack{M+1 \leqslant n \leqslant M+N \\ n \mid k}} c_{k/n},\ A_{-k} = \overline{A}_k\ (k \geqslant 1) \tag{9}$$

(an empty sum denoting zero).

By Parseval's theorem we then have:

$$\int_0^1 \Big| \sum_{n=M+1}^{M+N} g(nx) \Big|^2 dx = 2 \sum_{k=1}^{\infty} |A_k|^2. \tag{10}$$

Now by (9)

$$\sum_{k=1}^{\infty} |A_k|^2 = \sum_{k=1}^{\infty} \Big| \sum_{\substack{M+1 \leqslant n \leqslant M+N \\ n \mid k}} c_{k/n} \Big|^2 =$$

$$\sum_{k=1}^{\infty} \Big(\sum_{\substack{M+1 \leqslant m \leqslant M+N \\ M+1 \leqslant n \leqslant M+N \\ m \mid k,\ n \mid k}} c_{k/m} \cdot \overline{c}_{k/n} \Big),$$

where the inner sum is to be extended over all couples m, n of divisors of k which fall in the segment $M+1 \leqslant u \leqslant M+N$. Now each couple of positive integers $\bar{u}$, $\bar{v}$ with common divisor d, such that $\bar{u} = ud$, $\bar{v} = vd$, can be written in the form k/m, k/n with $k = \sigma uvd$, $m = \sigma v$, $n = \sigma u$.

Therefore we find

$$\sum_{k=1}^{\infty} |A_k|^2 = \sum_{d=1}^{\infty} \sum_{(u,v)}^{*} c_{ud} \cdot \overline{c}_{vd}, \tag{11}$$

where $\sum\limits_{(u,v)}^{*}$ is to be extended over all triples of positive integers u, v, σ, for which

$$(u, v) = 1,\ M+1 \leqslant \sigma u \leqslant M+N,\ M+1 \leqslant \sigma v \leqslant M+N.$$

If $u = v$ ($= 1$), the number σ runs through the N values, $M+1$, $M+2, \ldots, M+N$. If $u \neq v$, it is clear that $1 \leqslant \sigma \leqslant N-1$, as our segment has the length $N-1$ whereas always $|u - v| \geqslant 1$ and therefore $|\sigma u - \sigma v| \geqslant \sigma$. Collecting

the couples u, v which belong to the same value of σ, we easily see, that the right-hand side of (11) is equal to $R_0N+\Omega(M, N)$, which proves the lemma, because of (10).

Remark. From (9) we deduce by means of the inequality of Cauchy-Schwarz-Riemann:

$$|A_k|^2 \leqslant d^*(k) \sum_{\substack{M+1\leqslant n\leqslant M+N \\ n|k}} |c_{k/n}|^2,$$

where $d^*(k)$ denotes the number of divisors of k lying in the segment $(M+1, M+N)$. Hence $d^*(k) \leqslant \max(N, d(k))$. Thus we find

$$\sum_{k=1}^{\infty} |A_k|^2 \leqslant \sum_{h=1}^{\infty} |c_h|^2 \sum_{n=M+1}^{M+N} d^*(nh),$$

hence if $p > 1$

$$\sum_{k=1}^{\infty} |A_k|^2 \leqslant \Big(\sum_{n=M+1}^{M+N} d(n)\Big) \sum_{h=1}^{p} |c_h|^2 d(h)$$
$$+N^2 \sum_{h=p+1}^{\infty} |c_h|^2 = O(N(M+N)^\epsilon p^\epsilon + N^2 R_p),$$

for any fixed value of $\epsilon > 0$. Choosing $p = N$ we find by (10):

$$\int_0^1 \Big|\sum_{n=M+1}^{M+N} g(nx)\Big|^2 dx = O(N^{1+\epsilon}(M+N)^\epsilon + N^2 R_N).$$

Using this relation and assuming (3) it is possible to prove that (5) holds almost everywhere in x. Thus we have got a simple proof of the special case $f_n(x) = nx$ of the general theorem mentioned above.

As, however, we want to prove (5) under the weaker condition (4) instead of (3), we must handle the sum $\sum_{k=1}^{\infty} |A_k|^2$ more carefully.

LEMMA 2. *Let $g(x) \in L^2$ denote a function with Fourier series* (6) *and let $\Omega(M, N)$ be defined by* (8). *If for $M \geqslant 0$, $N \geqslant 1$ we put*

$$\Lambda(M, N) = 2 \sum_{d=1}^{\infty} \sum_{\sigma=1}^{N-1} \sum_{\substack{(u,v)=1 \\ M+1\leqslant \sigma u<\sigma v\leqslant M+N}} |c_{ud}||c_{vd}|, \qquad (12)$$

we have

$$\Omega(M, N) \leqslant \Lambda(M, N) \quad (M \geqslant 0, N \geqslant 1). \tag{13}$$

Putting

$$\Lambda^*(N) = 2N \sum_{d=1}^{\infty} \sum_{\substack{(u,v)=1 \\ 1 \leqslant u < v \leqslant N}} \frac{|c_{ud}|\,|c_{vd}|}{v} \tag{14}$$

and

$$\Lambda^{**}(N) = N \sum_{d=1}^{\infty} \sum_{u=1}^{2N} |c_{ud}|^2 \tag{15}$$

we have

$$\Lambda(0, N) \leqslant \Lambda^*(N), \tag{16}$$

$$\Lambda(N, N) \leqslant 2\Lambda^*(N) + \Lambda^{**}(N). \tag{17}$$

Proof. Formula (13) is trivial by (12) and (8). To prove (16) we only need to remark that for $M=0$ the right-hand side of (12) equals

$$2 \sum_{d=1}^{\infty} \sum_{\substack{(u_1, v_1)=1 \\ 1 \leqslant u_1 < v_1 \leqslant N}} |c_{u_1 d}|\,|c_{v_1 d}| \sum_{\sigma=1}^{[N/v_1]} 1.$$

To prove (17) we consider the inner sum of (12), putting $M = N$. Collecting all terms for which $v \leqslant N$, we remark that for these terms automatically $\sigma \leqslant 2N/v$. Collecting the remaining terms, i.e. the terms for which $v > N$, we remark that the corresponding σ for such a term necessarily equals 1. Corresponding to those two groups of terms we may write

$$\Lambda(N, N) \leqslant 4N \sum_{d=1}^{\infty} \sum_{\substack{(u,v)=1 \\ 1 \leqslant u < v \leqslant N}} \frac{|c_{ud}|\,|c_{vd}|}{v} + 2 \sum_{d=1}^{\infty} \sum_{\substack{(u,v)=1 \\ N+1 \leqslant u < v \leqslant 2N}} |c_{ud}|\,|c_{vd}|$$

$$\leqslant 2\Lambda^*(N) + \sum_{d=1}^{\infty} \Big\{ \sum_{N+1 \leqslant u \leqslant 2N} |c_{ud}| \Big\}^2$$

$$\leqslant 2\Lambda^*(N) + N \sum_{d=1}^{\infty} \sum_{N+1 \leqslant u \leqslant 2N} |c_{ud}|^2 \leqslant 2\Lambda^*(N) + \Lambda^{**}(N).$$

Lemma 3. *If* $\Lambda^*(N)$ *denotes the expression* (14) *of Lemma* 2, *and if* (4) *holds, the series*

$$\sum_{N=1}^{\infty} \frac{\log^2 N}{N^3} \Lambda^*(N)$$

converges.

[With respect to the final form of the lemma, I am indebted to Mr. J. G. Van der Corput for a joint discussion of the best way of handling $\Lambda^*(N)$].

PROOF. By (14) we have:

$$\sum_{N=1}^{\infty} \frac{\log^2 N}{N^3} \Lambda^*(N) = 2 \sum_{d=1}^{\infty} \sum_{N=1}^{\infty} \frac{\log^2 N}{N^2} \sum_{\substack{(u,v)=1 \\ 1 \leqslant u < v \leqslant N}} \frac{|c_{ud}|\,|c_{vd}|}{v}$$
$$= 2 \sum_{d=1}^{\infty} \sum_{\substack{(u,v)=1 \\ 1 \leqslant u < v < \infty}} \frac{|c_{ud}|\,|c_{vd}|}{v} \sum_{N=v}^{\infty} \frac{\log^2 N}{N^2},$$

in case of divergence all sums denoting infinity. Now

$$\sum_{N=v}^{\infty} \frac{\log^2 N}{N^2} \leqslant \frac{\log^2 v}{v^2} + \int_v^{\infty} \frac{\log^2 y \, dy}{y^2} = \frac{\log^2 v}{v^2} + \frac{\log^2 v}{v} + 2\frac{\log v}{v}$$
$$+ \frac{2}{v} \leqslant 8\,\frac{\log^2 v}{v}.$$

Hence

$$\sum_{N=1}^{\infty} \frac{\log^2 N}{N^3} \Lambda^*(N) \leqslant 16 \sum_{d=1}^{\infty} \sum_{\substack{(u,v)=1 \\ 1 \leqslant u < v < \infty}} \frac{|c_{ud}|\,|c_{vd}| \log^2 v}{v^2}$$

$$\leqslant 8 \sum_{d=1}^{\infty} \sum_{\substack{(u,v)=1 \\ 1 \leqslant u < v < \infty}} \frac{|c_{ud}|^2 + |c_{vd}|^2}{v^2} \log^2 v$$

$$\leqslant 8 \sum_{d=1}^{\infty} \sum_{u=1}^{\infty} |c_{ud}|^2 \sum_{v=u+1}^{\infty} \frac{\log^2 v}{v^2} + 8 \sum_{d=1}^{\infty} \sum_{v=1}^{\infty} \frac{|c_{vd}|^2 \log^2 v}{v}$$

$$\leqslant 72 \sum_{d=1}^{\infty} \sum_{a=1}^{\infty} \frac{|c_{ad}|^2 \log^2 a}{a} = 72 \sum_{\nu=1}^{\infty} |c_\nu|^2 \sum_{\delta \mid \nu} \frac{\log^2 \delta}{\delta} < \infty,$$

because of (4).

LEMMA 4. *If* $\Lambda^{**}(N)$ *denotes the expression* (15) *of Lemma* 2 *and if* (4) *holds, the series*

$$\sum_{N=1}^{\infty} \frac{\log^2 N}{N^3} \Lambda^{**}(N)$$

converges.

PROOF. By (15) we have

$$\sum_{N=1}^{\infty} \frac{\log^2 N}{N^3} \Lambda^{**}(N) = \sum_{d=1}^{\infty} \sum_{N=1}^{\infty} \frac{\log^2 N}{N^2} \sum_{u=1}^{2N} |c_{ud}|^2$$

$$= \sum_{d=1}^{\infty} \sum_{u=1}^{\infty} |c_{ud}|^2 \sum_{u/2 \leqslant N < \infty} \frac{\log^2 N}{N^2} \leqslant 16 \sum_{d=1}^{\infty} \sum_{u=2}^{\infty} \frac{|c_{ud}|^2 \log^2 u}{u}$$

$$+2 \sum_{d=1}^{\infty} |c_d|^2 = 16 \sum_{\nu=1}^{\infty} |c_\nu|^2 \sum_{\delta | \nu} \frac{\log^2 \delta}{\delta} + 2 \sum_{d=1}^{\infty} |c_d|^2 < \infty,$$

because of (4).

We finally make use of the following theorem, which has been proved by Mr I. S. Gál and the author [2]:

LEMMA 5. *Let S denote a measurable part of an Euclidean space, and let $F(M, N; x) \in L^p$ in S for a given $p \geqslant 1$ $(M = 0, 1, \ldots; N = 1, 2, \ldots; x \in S)$. Assume that*
$F(M, 0; x) = 0$, $F(M, N; x) \geqslant 0$, and for $0 \leqslant N' \leqslant N$,

$$F(M, N; x) \leqslant F(M, N'; x) + F(M+N', N-N'; x), \tag{18}$$

and (C being a positive constant)

$$\int_S F(M, N; x)^p \, dx \leqslant C \Psi(N) \phi^*(M, N) \quad (M \geqslant 0, N \geqslant 1), \tag{19}$$

where $\Psi(N)/N$ is a non-decreasing function, whereas $\phi^*(M, N) \geqslant 1$. Let $\psi(N) > 0$ denote a non-decreasing function, such that

$$\psi(2^n) \geqslant \phi^*(0, 2^n) + \sum_{\lambda=1}^{n} \sum_{\mu=0}^{2^{n-\lambda}-1} 2^{\lambda-n} \phi^*(2^n + \mu 2^\lambda, 2^{\lambda-1}) \tag{20}$$

and let $\chi(N)$ denote a non-decreasing positive function, satisfying $\sum_{N=1}^{\infty} (N\chi(N))^{-1} < \infty$, then almost everywhere in S:

$$F(0, N; x) = o\left(\Psi(N) \psi(N) \chi(N) (\log N)^{p-1}\right)^{1/p}.$$

3. Proof of Theorem 1.

We shall apply Lemma 5. Put $p = 2$ and

$$F(M, N; x) = \Big| \sum_{n=M+1}^{M+N} g(nx) \Big| .$$

It is clear, that (18) is satisfied, if S denotes the segment $0 \leqslant x \leqslant 1$. Further put

$$\Psi(N) = N, \; \phi^*(M, N) = 1 + \Omega(M, N)/N. \tag{21}$$

Then (19) is satisfied with $C = \max(2 R_0, 2)$, because of Lemma 1. We finally put

$$\psi(N) = \frac{9\Lambda^*(N) + 4\Lambda^{**}(N)}{N} \log N + 2 \log N + 1$$

and

$$\chi(N) = \frac{N}{\psi(N) \log N}. \tag{22}$$

From (14) and (15) we see at once that $\psi(N)$ is a positive and non-decreasing function. Further we have

$$\sum_{N=1}^{\infty} \frac{1}{N \;\; (N)} = \sum_{N=1}^{\infty} \frac{\psi(N) \log N}{N^2} < \infty$$

because of Lemma 3, Lemma 4 and (22).

In order to prove the inequality (20), we remark that by (21) and (13)

$$\begin{aligned} &\phi^*(0, 2^n) + \sum_{\lambda=1}^{n} \sum_{\mu=0}^{2^{n-\lambda}-1} 2^{\lambda-n} \phi^*(2^n + \mu 2^\lambda, 2^{\lambda-1}) \\ &\leqslant 1 + \frac{1}{2^n} \Lambda(0, 2^n) + \sum_{\lambda=1}^{n} \sum_{\mu=0}^{2^{n-\lambda}-1} \Big(1 + \frac{1}{2^{\lambda-1}} \Lambda(2^n + \mu 2^\lambda, 2^{\lambda-1})\Big) 2^{\lambda-n} \\ &\leqslant 1 + \frac{1}{2^n} \Lambda^*(2^n) + n + \sum_{\lambda=1}^{n} 2^{-n+1} \sum_{\mu=0}^{2^{n-\lambda}-1} \Lambda(2^n + \mu 2^\lambda, 2^{\lambda-1}), \end{aligned} \tag{23}$$

by (16). The expression $\Lambda(2^n + \mu 2^\lambda, 2^{\lambda-1})$ is defined by (12). We remark that for two different values of $\mu = 0, 1, \ldots, 2^{n-\lambda} - 1$, the corresponding segments $2^n + \mu 2^\lambda + 1 \leqslant x \leqslant 2^n + \mu 2^\lambda + 2^{\lambda-1}$ have no points in common, and are all contained in the segment $2^n + 1 \leqslant x \leqslant 2^{n+1}$. Therefore it is obvious that

$$\sum_{\mu=0}^{2^{n-\lambda}-1} \Lambda(2^n+\mu 2^\lambda, 2^{\lambda-1}) \leqslant \Lambda(2^n, 2^n) \leqslant 2\Lambda^*(2^n)+\Lambda^{**}(2^n),$$

by (17). Hence the left-hand side of (23) is

$$\leqslant 1+n+\frac{1}{2^n}\Lambda^*(2^n)+n.2^{-n+1}\left\{2\Lambda^*(2^n)+\Lambda^{**}(2^n)\right\}$$

$$\leqslant 1+2\log(2^n)+\frac{9\Lambda^*(2^n)+4\Lambda^{**}(2^n)}{2^n}\log(2^n)=\psi(2^n)$$

because of (22).

Now, all the conditions of Lemma 5 being fulfilled, we can apply this lemma, and we find by (21) and (22):

$$F(0, N; x) = \left|\sum_{n=1}^{N} g(nx)\right| = o(\Psi(N)\psi(N)\chi(N)\log N)^{\frac{1}{2}} = o(N)$$

almost everywhere.

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REMARKS ON GAUSSIAN SUMS AND TAUBERIAN THEOREMS

By S. BOCHNER

Kronecker's derivation of the reciprocity formula for certain Gaussian sums from the modular relation

$$\tfrac{1}{2}+\sum_{n=1}^{\infty} e^{-\pi n^2/x} = x^{\frac{1}{2}}\left\{ \tfrac{1}{2}+\sum_{n=1}^{\infty} e^{-\pi n^2 x} \right\} \tag{1}$$

is fundamentally an Abelian-Tauberian argument (see the (not very rigorous) discussion in [1, p. 1781]), and we will reproduce his argument in a general set-up, although we have no specific instances to offer other than those well known.

For complex numbers c_n and non-negative exponents ν_n, $\nu_n \to \infty$, and a

$$\gamma > 0 \tag{2}$$

we set up for a real number ρ the Abelian limit

$$A_\gamma(c_n, \nu_n; \rho) = \lim_{\varepsilon \to 0} \varepsilon^\gamma \sum_n c_n e^{i\nu_n \rho} e^{-\varepsilon \nu_n} \tag{3}$$

and the Cesaro limit

$$G_\gamma(c_n, \nu_n; \rho) = \lim_{\nu \to \infty} \frac{\Gamma(\gamma+1)}{\nu^\gamma} \sum_{\nu_n < \nu} c_n e^{i\nu_n \rho} \tag{4}$$

and we record the following statements [cf. 3, Chr. VI]:

LEMMA. *If the limit* (4) *exists, then the limit* (3) *exists, and they are equal. Conversely, if the limit* (3) *exists and*

$$\sum_{\nu \leqslant \nu_n < \theta \nu} |c_n| = O(\nu^\gamma), \quad \nu \to \infty \tag{5}$$

for some $\theta > 0$, *no matter how large, then the limit* (4) *exists.*

The lemma will be used for proving the following proposition.

THEOREM I. *If we are given in the half-plane*

$$Re\{x\} > 0 \tag{6}$$

for some

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$$\delta > 0 \tag{7}$$

a relation (for general relations of the form (8) *see* [2])

$$\sum_0^\infty b_n e^{-\mu_n/x} = x^\delta \sum_0^\infty a_n e^{-\lambda_n x} \tag{8}$$

with $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \to \infty$; $0 < \mu_0 < \mu_1 < \mu_2 < \cdots \to \infty$; *if for some* γ, *with* $\gamma \geqslant \delta$, *but preferably with (we are not excluding* $\gamma \neq \delta$ *because of the results in* [4], *cf.* [7, *pp.* 110-113])

$$\gamma = \delta, \tag{9}$$

[*if the limit* (3) *or* (4) *exists for* γ_0 *it also exists for* $\gamma > \gamma_0$ *and has value zero*] *the limit* $G_\gamma(a_n, \lambda_n; \rho)$ *exists for some real* $\rho \neq 0$; *and if the condition*

$$\sum_{\mu \leqslant \mu_n < \theta\mu} |b_n| = O(\mu^\gamma), \quad \mu \to \infty \tag{10}$$

is satisfied; then $G_\gamma(b_n, \mu_n; -1/\rho)$ *likewise exists and we have the reciprocity formula*

$$|\rho|^{2\gamma} G_\gamma(b_n, \mu_n; -1/\rho) = (-i\rho)^\delta G_\gamma(a_n, \lambda_n; \rho), \tag{11}$$

where $(-i\rho)^\delta = \lim_{\epsilon \to 0} (\epsilon - i\rho)^\delta$.

More generally, if we are given a relation

$$\sum_0^\infty b_n \exp\left(-\mu_n \frac{ax - ib}{icx + d}\right) = (cx - id)^\delta \sum_0^\infty a_n e^{-\lambda_n x} \tag{12}$$

with real numbers a, b, c, d for which

$$ad - bc = 1, \tag{13}$$

then under the same assumption we have

$$|c + d|^{2\gamma} G_\gamma\left(b_n, \mu_n; \frac{a\rho + b}{c\rho + d}\right) = (-i(c\rho + d))^\delta G_\gamma(a_n, b_n; \rho), \tag{14}$$

where $(-i(c\rho + d))^\delta = \lim_{\epsilon \to 0} (c\epsilon - i(c\rho + d))^\delta$; and in particular ($\rho = 0$) we have

$$|d|^{2\gamma} G_\gamma(b_n, \mu_n; b/d) = (-i\,d)^\delta G_\gamma(a_n, \lambda_n; 0). \tag{15}$$

Corollary. If we consider the original relation (1) and if we denote the corresponding expression $\pi^{1/2} G_{\frac{1}{2}}(1, \pi n^2; \rho)$ by $G(\rho)$, then due to

$$2(s+n)^2 \equiv 2n^2 \pmod{2s}$$
$$(2s-n)^2 \equiv n^2 \pmod{2s}$$

it can be easily seen that for a rational number

$$\rho = r/s \tag{16}$$

the limit $G(\rho)$ actually exists and has the value

$$G\left(\frac{r}{s}\right) = \frac{1}{|s|}\sum_{n=1}^{|s|} e^{\pi i n^2 r/s}, \tag{17}$$

and the conclusion from Theorem 1 is as follows:

For rational $\rho \neq 0$, and say $\rho > 0$, we have

$$G(-1/\rho) = e^{-\pi i/4}\rho^{-\frac{1}{2}}G(\rho), \tag{18}$$

and for any integer matrix

$$\begin{pmatrix} a, & b \\ c, & d \end{pmatrix} \equiv \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix} \pmod 2 \tag{19}$$

we have

$$|c\rho+d|. \quad G\left(\frac{a\rho+b}{c\rho+d}\right) = \lim_{\varepsilon\to 0}\,(c\varepsilon - i(c\rho+d))^{\frac{1}{2}}G(\rho). \tag{20}$$

In particular, if two integers b, d can be supplemented to a matrix (19), then we have

$$|d|\,G(b/d) = \lim_{\varepsilon\to 0}\,(c\varepsilon - id)^{\frac{1}{2}} \tag{21}$$

that is

$$\sum_{n=1}^{|d|} e^{\pi i n^2 b/d} = \lim_{\varepsilon\to 0}\,(c\varepsilon - id)^{\frac{1}{2}}. \tag{22}$$

The simplest way of proving Theorem 1 is to interpose a theorem about the Abelian averages (3).

THEOREM 2. *Under the assumption*

$$\sum_n \mu_n\,|\,b_n\,|\,e^{-\varepsilon\mu_n} = o(\varepsilon^{-\gamma-2}), \tag{23}$$

if we are given relation (8) *then we have*

$$|\rho|^{2\gamma}A_\gamma(b_n, \mu_n; -1/\rho) = (-i\rho)^\delta A_\gamma(a_n, \lambda_n; \rho) \tag{24}$$

whenever the right side limit exists; and if we are given relation (12) *then we have*

$$|c\rho+d|^{2\gamma}A_\gamma\left(b_n, \mu_n; \frac{a\rho+b}{c\rho+d}\right) = (-i(c\rho+d))^\delta A_\gamma(a_n, \lambda_n; \rho). \tag{25}$$

PROOF OF THEOREM 2. In case of relation (8) if we put $x=\epsilon-i\rho$ and multiply both sides of it by ϵ^γ, then the right side converges towards the right side of (24) and the joint value is then

$$\lim_{\epsilon\to\infty} \epsilon^\gamma \sum_n b_n e^{-\mu_n/(\epsilon-i\rho)} \tag{26}$$

the limit existing. Now, in order to identify this with the left side of (24) we note that

$$\frac{1}{\epsilon-i\rho}=\frac{i}{\rho}+\frac{\epsilon}{\rho^2}-\frac{\epsilon^2}{\rho^2(\epsilon-i\rho)}=\frac{i}{\rho}+\frac{\epsilon}{\rho^2}+O(\epsilon^2)$$

so that (26) is

$$\lim_{\epsilon\to 0} \epsilon^\gamma \sum b_n e^{-\frac{i\mu_n}{x}-\mu_n(\epsilon/\rho^2+O(\epsilon^2))}$$

$$=|\rho|^{2\gamma}\lim_{\epsilon\to 0} \epsilon^\gamma \sum b_n e^{-\frac{i\mu_n}{x}-\mu_n(\epsilon+O(\epsilon^2))},$$

and it suffices to show that

$$\epsilon^\gamma \sum_n |b_n| e^{-\epsilon\mu_n} |. | 1-e^{O(\epsilon^2\mu_n)}|=o(1). \tag{27}$$

However, because of $|e^z-1|\leqslant |z|e^{|z|}$, the left side of (27) is majorized by

$$O(\epsilon^{\gamma+2})\sum_n \mu_n |b_n| e^{-\mu_n(\epsilon-K\epsilon^2)}$$

for some $K>0$. If we replace ϵ by $\epsilon-K\epsilon^2$ this becomes

$$O(\epsilon^{\gamma+2})\sum_n \mu_n |b_n| e^{-\epsilon\mu_n}$$

and this is indeed $o(1)$ by explicit assumption (23). For the more general relation (12) the more general identity

$$\frac{a\epsilon-i(a\rho+b)}{ic\epsilon+(c\rho+d)}=-i\frac{a\rho+b}{c\rho+d}+\frac{\epsilon}{(c\rho+d)^2}+O(\epsilon^2)$$

leads to the same conclusion.

PROOF OF THEOREM 1. It is easily seen that the Tauberian assumption (10) implies the Tauberian assumption (23),— it actually implies $O(\epsilon^{-\gamma-1})$ —, so that first of all the conclusion of Theorem 2 obtains. But assumption (10) also validates the lemma and this then gives the conclusion of Theorem 1.

Several variables. As regards multidimensional extensions, two distinct possibilities suggest themselves. The first, which in this connection was brought to the fore (though was not originally introduced) by E. Hecke, takes in a k-fold octant

$$Re\{x_j\}, j = 1, \ldots, k \tag{28}$$

a relation

$$\sum_n b_n e^{-\Sigma_j \mu_{n_j}/x_j} = x_1^{\delta_1} \ldots x_k^{\delta_k} \sum_n a_n e^{-\Sigma_j \lambda_{n_j} x_j} \tag{29}$$

with $\lambda_{n_j} \geqslant 0,\ \mu_{n_j} \geqslant 0,\ \delta_j > 0,$

puts

$$x_j = \epsilon - i\rho_j \tag{30}$$

and lets $\epsilon \to 0$. The resulting extension of (11) is

$$\lim_{\mu\to\infty} \frac{1}{\mu^\gamma} \sum_{\Sigma \mu_{n_j}/\rho_j \leqslant \mu} b_n e^{-i\Sigma_j \mu_{nj}/\rho_j} \tag{31}$$

$$= \Pi_j (-i\rho_j)^{\delta_j} \lim_{\lambda\to\infty} \frac{1}{\lambda^\gamma} \sum_{\Sigma_j \lambda_{n_j}} a_n e^{i\Sigma_j \lambda_{n_j}\rho_j}$$

preferably for

$$\gamma = \delta_1 + \cdots + \delta_k, \tag{32}$$

and it is valid whenever the right side limit exists and the assumption

$$\sum_{\mu \leqslant \Sigma_j \mu_{n_j} \leqslant \theta\mu} |b_n| = O(\mu^\gamma),\ \mu \to \infty \tag{33}$$

is fulfilled. More generally we might try to put

$$x_j = \sigma_j (\varepsilon - i\rho_j)$$

for non-equal $\sigma_j > 0$, but this can be handled in the following manner. First put $x_j = \sigma_j y_j$ which transforms (29) into a similar relation, and in the latter put $y_j = \epsilon - i\rho_j$ as before, and derive the relation as before.

In Hecke's case [5] corresponding to an ideal class in an algebraic number field there is a theta relation which is associated with it in the same manner as (1) is with the basic field of rationals, and for suitable choices of (conjugate) numbers σ_j, ρ_j, the two sides in (31) will again be finite sums. They are generalizations of the

Gaussian sums, and the identity (31) is expressive of a reciprocity law in the field. The same theta function probably also satisfies relations analogous to (12) with fractional expressions

$$\frac{a_j x - i b_j}{i c_j x + d_j}$$

in which the constants are suitable conjugate numbers and there probably are "evaluating" formulas generalizing (15) but they apparently have been nowhere written out.

Matrix Variables. [8]. Another possibility suggesting itself is the generalization of the symbols λ_n, μ_n, x to k by k matrices each, say symmetric ones, positive semi-definite or positive definite respectively, with matrix traces for "inner products" thus :

$$\sum_0^\infty b_n e^{-\mathrm{trace}\,(\mu_n x^{-1})} = (\det|x|)^\delta \sum_0^\infty a_n e^{-\mathrm{trace}\,(\lambda_n x)}.$$

If now we put $x = \varepsilon 1 - i\rho$ where 1 is the unit matrix, ρ is any non-singular real matrix and ε is a scalar, then on letting $\varepsilon \to 0$ we obtain, preferably for

$$\gamma = k\delta,$$

the reciprocity formula

$$\lim_{\mu\to\infty} \frac{1}{\mu^\gamma} \sum_{\mathit{trace}\,(\mu_n \rho^{-2})<\mu} b_n e^{-i\,\mathit{trace}\,(\mu_n \rho^{-1})}$$

$$= (\mathit{det}\,|-i\rho\,|)^\delta \lim_{\gamma\to\infty} \frac{1}{\lambda^\gamma} \sum_{\mathit{trace}\,\lambda_n<\lambda} a_n e^{i\,\mathit{trace}\,(\lambda_n \rho)},$$

with condition (33) being replaced by

$$\sum_{\mu \leqslant \mathit{trace}\,\mu_n < \theta\mu} |b_n| = O(\mu^\gamma), \quad \mu \to \infty.$$

More generally, if in analogy to (12) we are given a relation

$$\sum_0^\infty b_n \exp\,(-\mathit{trace}\,[\mu_n(ax-ib)(icx+d)^{-1}])$$

$$= (\det|x|)^{\delta} \sum_{0}^{\infty} a_n e^{-\operatorname{trace}(\lambda_n x)},$$

with suitable matrices a, b, c, d for which a condition corresponding to (13) is satisfied among others, then we are led to the reciprocity formula

$$\lim_{\mu\to\infty} \frac{1}{\mu^{\gamma}} \sum_{\operatorname{trace}(\mu_n(\rho c+d)^{-2})<\mu} b_n \exp\left\{ i \operatorname{trace}[\mu_n(a\rho+b)(c\rho+d)^{-1}] \right\}$$

$$= (\det|-i(c\rho+d)|)^{\delta} \lim_{\lambda\to\infty} \frac{1}{\lambda^{\gamma}} \sum_{\operatorname{trace}(\lambda_n)<\lambda} a_n\, e^{i \operatorname{trace}(\lambda_n \rho)},$$

but it is not known to us whether particular finite sum instances of this formula have been envisaged.

Addendum. This addition will not affect the general theorem stated but only one of the conclusions drawn. For the Gaussian averages

$$G\left(\frac{r}{s}\right) = \frac{1}{2|s|} \sum_{n=1}^{2|s|} e^{\pi i n^2 r/s}$$

we first of all had the classical inversion formula

$$|\rho|\, G\left(-\frac{1}{\rho}\right) = \lim_{\varepsilon\to 0} (\varepsilon - i\rho)^{\frac{1}{2}} G(\rho), \tag{18}$$

and we now observe that in it the square root is to be determined from

$$-\frac{\pi}{2} < \operatorname{Im} \; [\log(\varepsilon - i\rho)] < \frac{\pi}{2}.$$

However in the subsequent relation

$$|c\rho+d|.\; G\left(\frac{a\rho+b}{c\rho+d}\right) = \lim_{\varepsilon\to 0} (c\varepsilon - i(c\rho+d))^{\frac{1}{2}} G(\rho) \tag{20}$$

the square root is not normalized in such an obvious manner, and is in fact given to within an eighth root of unity $\eta(a, b, c, d)$ whose explicit determination is a recondite problem; so that the conclusions (21), (22) that were drawn from (20), although not incorrect, are not enlightening. Nevertheless, if we envisage relation (20) for

two different rational numbers ρ, σ,—of which one could be permitted to be zero—then the eighth root can be eliminated thus:

$$\lim_{\varepsilon\to\infty} (c\varepsilon - i\,(c\sigma+d))^{\frac{1}{2}} |c\rho+d|\; G(\sigma)\; G\left(\frac{a\rho+b}{c\rho+d}\right)$$

$$= \lim_{\varepsilon\to 0} (c\varepsilon - i\,(c\rho+d))\, |c\sigma+d|.\; \dot{G}(\rho)\; G\left(\frac{a\sigma+b}{c\sigma+d}\right),$$

and we are recording this relation only because we have not noticed it included in monographs dealing with the topic.

We also note that the unimodular matrices admissible in the last relation are those for which

$$\begin{pmatrix} a, & b \\ c, & d \end{pmatrix} \equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or} \equiv \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix} \pmod 2 \qquad (19)$$

and that in formula (19) as originally stated the last matrix was omitted inadvertently.

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ON GROUPS OF HOMOTHETIC TRANSFORMATIONS IN RIEMANNIAN SPACES

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1. We consider an infinitesimal transformation

$$\bar{x}^\lambda = x^\lambda + \xi^\lambda(x)dt \qquad (\lambda, \mu, \nu, \ldots = 1, 2, \ldots, n) \tag{1.1}$$

in a Riemannian space, dt being an infinitesimal. Then the Lie derivative $Xg_{\mu\nu}$[6] of the fundamental tensor $g_{\mu\nu}$ of the space with respect to (1.1) is given by

$$Xg_{\mu\nu} = \xi^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} + \frac{\partial \xi^\alpha}{\partial x^\mu} g_{\alpha\nu} + \frac{\partial \xi^\alpha}{\partial x^\nu} g_{\mu\alpha} = \xi_{\mu;\,\nu} + \xi_{\nu;\,\mu}, \tag{1.2}$$

where the semi-colon denotes the covariant derivative and $\xi_\mu = g_{\mu\alpha}\xi^\alpha$, and the Lie derivative $X\left\{ {\lambda \atop \mu\nu} \right\}$ of Christoffel symbols $\left\{ {\lambda \atop \mu\nu} \right\}$ by

$$\begin{aligned} X\left\{ {\lambda \atop \mu\nu} \right\} &= \frac{\partial^2 \xi^\lambda}{\partial x^\mu \partial x^\nu} + \xi^\alpha \frac{\partial \left\{ {\lambda \atop \mu\nu} \right\}}{\partial x^\alpha} - \frac{\partial \xi^\lambda}{\partial x^\alpha} \left\{ {\alpha \atop \mu\nu} \right\} + \frac{\partial \xi^\alpha}{\partial x^\mu} \left\{ {\lambda \atop \alpha\nu} \right\} + \frac{\partial \xi^\alpha}{\partial x^\nu} \left\{ {\lambda \atop \mu\alpha} \right\} \\ &= \xi^\lambda{}_{;\,\mu;\,\nu} + R^\lambda_{\mu\nu\omega}\xi^\omega, \end{aligned} \tag{1.3}$$

where $R^\lambda_{\mu\nu\omega}$ is the curvature tensor.

If we use this notation, (*i*) the motion, (*ii*) the conformal transformation, (*iii*) the affine collineation, and (*iv*) the projective collineation in a Riemannian space will respectively be characterized by the following equations:

(*i*) $Xg_{\mu\nu} = 0$,

(*ii*) $Xg_{\mu\nu} = 2\phi g_{\mu\nu}$, where $\phi = [1/(2n)]\, g^{\beta\gamma}(Xg_{\beta\gamma})$,

(*iii*) $X\left\{ {\lambda \atop \mu\nu} \right\} = 0$,

(*iv*) $X\left\{ {\lambda \atop \mu\nu} \right\} = \delta^\lambda_\mu \varphi_\nu + \delta^\lambda_\nu \varphi_\mu$, where $\varphi_\nu = [1/(n+1)] X\left\{ {\alpha \atop \alpha\nu} \right\}$.

The present author believes that the idea of Lie derivative is a very powerful tool for the study of such transformations. Besides the above well-known ones, a homothetic transformation in a Riemannian space is

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defined as one by which the corresponding distances are always in the same constant ratio, and consequently, if we use the above notation, it is characterized by

$$Xg_{\mu\nu} = 2cg_{\mu\nu}, \tag{1.4}$$

where c is the so-called homothetic constant. If c is equal to zero, then (1.4) gives the motion. Thus we call proper homothetic transformation the one for which c is not equal to zero. Such transformations have been recently studied by B. Shanks[4] without the use of Lie derivatives.

The purpose of the present paper is to give some further results obtained by the use of Lie derivatives in the study of groups of homothetic transformations. Although it is not indispensable for the paper, we shall give in § 2 brief sketches of the proofs by use of Lie derivatives of the results of B. Shanks, in the hope that this will show how the idea of Lie derivative is fitted for such a study. In § 3, we state some results supplementary to those of § 2. In §§ 4 and 5, we deal with groups of homothetic transformations. In § 6, we consider the subgroups of homothetic transformations contained in groups of conformal transformations or in groups of affine collineations. Finally, we prove, in § 7, theorems corresponding to those of Fubini [1].

2. If we take a coordinate system in which $\xi^\lambda = \delta^\lambda_1$, then the equations (1.2) and (1.4) give $\partial g_{\mu\nu}/\partial x^1 = 2cg_{\mu\nu}$, from which, we have

$$g_{\mu\nu} = e^{2cx^1} f_{\mu\nu}(x^2, x^3, \ldots, x^n). \tag{2.1}$$

Conversely, if there exists a coordinate system in which the fundamental tensor takes the form (2.1), then the space admits a one-parameter group of homothetic transformations $\bar{x}^\lambda = x^\lambda + \delta^\lambda_1 . t$ with homothetic constant c. Thus we have

THEOREM 1. *If a Riemannian space admits an infinitesimal homothetic transformation, then it admits also a one-parameter group of homothetic transformations generated by the infinitesimal one.*

THEOREM 2. *A necessary and sufficient condition that a Riemannian space admit a one-parameter group of homothetic transformations with homothetic constant c is that there exist a coordinate system in which the fundamental tensor takes the form* (2.1).

If we take a coordinate system in which $\xi^\lambda = x^\lambda$, then the equations (1.2) and (1.4) will give $x^\alpha(\partial g_{\mu\nu}/\partial x^\alpha) = 2(c-1)\, g_{\mu\nu}$, from which we can see that $g_{\mu\nu}$ are homogeneous functions of degree $2(c-1)$ with respect to x^λ. Thus we have

THEOREM 3. *A necessary and sufficient condition that a Riemannian space admit a one-parameter group of homothetic transformations with homothetic constant c is that there exist a coordinate system in which the components of the fundamental tensor are homogeneous functions of degree $2(c-1)$ of the coordinates.*

The equation (1.4) and a formula on Lie derivative [6]

$$X\left\{ {\lambda \atop \mu\nu} \right\} = \tfrac{1}{2}\, g^{\lambda\alpha}\left[(Xg_{\alpha\mu})_{;\nu} + (Xg_{\alpha\nu})_{;\mu} - (Xg_{\mu\nu})_{;\alpha}\right] \qquad (2.2)$$

give $X\left\{ {\lambda \atop \mu\nu} \right\} = 0$. Thus a homothetic transformation is an affine collineation. Conversely, if a conformal transformation is an affine collineation, then we have $Xg_{\mu\nu} = 2\phi g_{\mu\nu}$ and $X\left\{ {\lambda \atop \mu\nu} \right\} = 0$, from which we can conclude that $\phi = \text{const.}$ Thus we have

THEOREM 4. *A necessary and sufficient condition that a transformation be a homothetic one is that it be a conformal one and at the same time an affine collineation.*

More generally, if a conformal transformation is a projective collineation, we have $Xg_{\mu\nu} = 2\phi g_{\mu\nu}$ and $X\left\{ {\lambda \atop \mu\nu} \right\} = \delta^\lambda_\mu \varphi_\nu + \delta^\lambda_\nu \varphi_\mu$. But (2.2) and these equations give $\phi = \text{const.}$, and $\varphi_\nu = 0$. Thus we have

THEOREM 5. *A necessary and sufficient condition that a transformation be a homothetic one is that it be a conformal one and at the same time a projective collineation.*

Now, the integrability conditions of the mixed system of partial differential equations:

$$(Xg_{\mu\nu} \equiv)\ \xi_{\mu\nu} + \xi_{\nu\mu} = (2/n)\ g^{\beta\gamma}\,\xi_{\beta\gamma} g_{\mu\nu}, \qquad (F_0)$$
$$\xi^{\lambda}{}_{;\mu} = \xi^{\lambda}{}_{\mu}\ (\equiv g^{\lambda\alpha}\,\xi_{\alpha\mu}),$$
$$(X\{{}^{\lambda}_{\mu\nu}\} \equiv)\ \xi^{\lambda}{}_{\mu;\nu} + R^{\lambda}{}_{\mu\nu\omega}\,\xi^{\omega} = 0,$$

being

$$XR^{\lambda}{}_{\mu\nu\omega} \equiv \xi^{\alpha} R^{\lambda}{}_{\mu\nu\omega;\alpha} - \xi^{\lambda}{}_{;\alpha} R^{\alpha}{}_{\mu\nu\omega} + \xi^{\alpha}{}_{;\mu} R^{\lambda}{}_{\alpha\nu\omega} + \xi^{\alpha}{}_{;\nu} R^{\lambda}{}_{\mu\alpha\omega} + \xi^{\alpha}{}_{;\omega} R^{\lambda}{}_{\mu\nu\alpha} = 0, \qquad (F_1)$$
$$XR^{\lambda}{}_{\mu\nu\omega;\sigma} \equiv \xi^{\alpha} R^{\lambda}{}_{\mu\nu\omega;\sigma;\alpha} - \xi^{\lambda}{}_{;\alpha} R^{\alpha}{}_{\mu\nu\omega;\sigma} + \xi^{\alpha}{}_{;\mu} R^{\lambda}{}_{\alpha\nu\omega;\sigma} + \xi^{\alpha}{}_{;\nu} R^{\lambda}{}_{\mu\alpha\omega;\sigma} + \xi^{\alpha}{}_{;\omega} R^{\lambda}{}_{\mu\nu\alpha;\sigma} + \xi^{\alpha}{}_{;\sigma} R^{\lambda}{}_{\mu\nu\omega;\alpha} = 0, \qquad (F_2)$$

and so on, we have

THEOREM 6. *A necessary and sufficient condition that there exist a homothetic transformation in a Riemannian space is that there exist a positive integer N such that the equations (F_0), (F_1), (F_2), $\cdots$, (F_N) are consistent algebraically in ξ^{λ} and $\xi^{\lambda}{}_{;\mu}$ and all ξ^{λ} and $\xi^{\lambda}{}_{;\mu}$ satisfying these equations satisfy (F_{N+1}).*

The complete integrability condition of the mixed system is that $XR^{\lambda}{}_{\mu\nu\omega} = 0$ be identically satisfied by any ξ^{λ} and $\xi^{\lambda}{}_{;\mu}$ satisfying (F_0), the number of ξ^{λ} and $\xi^{\lambda}{}_{;\mu}$ which can be given arbitrarily being $n^2+n-[\frac{1}{2}\,n(n+1)-1] = \frac{1}{2}n(n+1)+1$. From this, we have $R^{\lambda}{}_{\mu\nu\omega} = \frac{R}{n(n-1)}\,(g_{\mu\nu}\,\delta^{\lambda}_{\omega} - g_{\mu\omega}\,\delta^{\lambda}_{\nu})$ and consequently, from (F_0) and (F_1), $XR^{\lambda}{}_{\mu\nu\omega} = 2cR^{\lambda}{}_{\mu\nu\omega} = 0$. Thus, we have

THEOREM 7. *In order that a Riemannian space admit a group of homothetic transformations of maximum order $\frac{1}{2}n(n+1)+1$, it is necessary and sufficient that the space be Euclidean.*

3. Now, if we consider an infinitesimal proper homothetic transformation whose trajectories are geodesics, then we have $(Xg_{\mu\nu} \equiv)\xi_{\mu;\nu}+\xi_{\nu;\mu} = 2cg_{\mu\nu}$ and $\xi_{\mu;\nu}\xi^{\nu} = \alpha\xi_{\mu}$, where α is a scalar. Multiplying the latter by ξ_{μ} and contracting, we have $c = \alpha$ by virtue of the former. Next,

multiplying the former by ξ^ν and contracting, we have $\xi_\mu = \left(\frac{1}{2c}\xi_\nu\xi^\nu\right)_{;\mu}$, which shows that ξ_μ is a gradient vector. Thus $\xi_{\mu;\nu}$ being symmetric in μ, and ν, we have

$$\xi_{\mu;\nu} = cg_{\mu\nu}, \tag{3.1}$$

i.e. the vector ξ^λ must be a concurrent one [5]. The converse being evident, we have

THEOREM 8. *A necessary and sufficient condition that a Riemannian space admit an infinitesimal proper homothetic transformation whose trajectories are geodesics is that it admit a concurrent vector field.*

A necessary and sufficient condition that a Riemannian space admit a concurrent vector field being that there exist a coordinate system in which the fundamental form takes the form [5]

$$ds^2 = (dx^1)^2 + (x^1)^2 f_{jk}(x^2, x^3, \dots, x^n)dx^j dx^k \quad (j, k = 2, 3, \dots, n), \tag{3.2}$$

we have

THEOREM 9. *A necessary and sufficient condition that a Riemannian space admit an infinitesimal proper homothetic transformation whose trajectories are geodesics is that there exist a coordinate system in which the fundamental form takes the form* (3.2).

The present author showed that a necessary and sufficient condition that an infinitesimal transformation change any conformal circle into a conformal circle is that the transformation be a conformal one [6]. Thus, from Theorem 5, we have

THEOREM 10. *A necessary and sufficient condition that an infinitesimal transformation be a homothetic one is that it change any conformal circle into a conformal circle and any geodesic into a geodesic.*

If $Xf = \xi^\alpha(\partial f/\partial x^\alpha)$ is a generator of a one-parameter group of homothetic transformations, then we have $Xg_{\mu\nu} = 2cg_{\mu\nu}$, $XR^\lambda_{\mu\nu\omega} = 0$ and consequently $XR_{\mu\nu} = 0$, $R_{\mu\nu}$ being the Ricci tensor. Thus if the space is an Einstein

space (or a space of constant curvature), i.e. if $R_{\mu\nu} = Rg_{\mu\nu}/n$, R being a constant, then we have $XR_{\mu\nu} = R(Xg_{\mu\nu})/n = 2cRg_{\mu\nu}/n = 0$. Thus if $R \neq 0$, then c must be zero, and consequently we have

THEOREM 11. *If an Einstein space (or a space of constant curvature) with non-vanishing scalar curvature admits a homothetic transformation, it is a motion, and consequently, an Einstein space (or a space of constant curvature) with non-vanishing scalar curvature cannot admit a proper homothetic transformation.*

4. From the equations which characterize the homothetic transformations, the following theorems are evident.

THEOREM 12. *If $\xi^\lambda_a(x)$ $(a, b, c, \ldots, f = 1, 2, \ldots, r)$ are r independent vectors defining r one-parameter groups of homothetic transformations in a Riemannian space, then $c^a\xi^\lambda_a$ is also a vector defining a homothetic one; c^a being constants not all zero.*

THEOREM 13. *If each of r generators of an r-parameter group of transformations generates a one-parameter group of homothetic transformations, then every transformation of the group is a homothetic transformation.*

If $X_a f = \xi^\lambda_a(\partial f/\partial x^\lambda)$ are generators of r one-parameter groups of transformations, then we have in general

$$(X_b X_c)g_{\mu\nu} \equiv (X_b X_c - X_c X_b)g_{\mu\nu} = X_{bc}g_{\mu\nu},$$

where

$$X_{bc}f = \left(\xi^\alpha_b \frac{\partial \xi^\lambda_c}{\partial x^\alpha} - \xi^\alpha_c \frac{\partial \xi^\lambda_b}{\partial x^\alpha} \right) \frac{\partial f}{\partial x^\lambda},$$

the operator X applied to the tensor $g_{\mu\nu}$ having always the meaning given by (1.2). On the other hand, if $X_a f$ are generators of r one-parameter groups of homothetic transformations, then we have $(X_b X_c)g_{\mu\nu} = 0$ and consequently $X_{bc}g_{\mu\nu} = 0$. Thus we have

THEOREM 14. *If $X_a f$ are generators of r one-parameter groups of homothetic transformations, then $X_{bc}f$ are those of one-parameter groups of motions.*

If $X_a f$ are r generators of an r-parameter group of transformations, then we have in general[6]

$$(X_b X_c) g_{\mu\nu} = c_{bc}{}^a X_a g_{\mu\nu},$$

$c_{bc}{}^a$ being constants of structure. But, for an r-parameter group of homothetic transformations, we have $(X_b X_c) g_{\mu\nu} = 0$, $X_a g_{\mu\nu} = 2c_a g_{\mu\nu}$ and consequently $c_{bc}{}^a c_a = 0$. Thus we have

THEOREM 15. *If $X_a f$ are r generators of an r-parameter group of homothetic transformations with homothetic constants c_a, then there exist relations $c_{bc}{}^a c_a = 0$ between the constants of structure and homothetic constants.*

In Theorem 6, a general solution being a linear combination with constant coefficients of a complete set of special solutions, from Theorem 14, we have

THEOREM 16. *If $X_a f$ are generators of the complete set of r one-parameter groups of homothetic transformations, they are generators of an r-parameter group G_r of homothetic transformations. Moreover, G_r must contain a complete set of one-parameter groups of motions, consequently, G_r contains a complete group of motions.*

5. If we assume that the space V_n admits an r-parameter group G_r of homothetic transformations whose generators are $X_a f = \xi_a^\lambda (\partial f/\partial x^\lambda)$, then we have $X_a g_{\mu\nu} = 2c_a g_{\mu\nu}$. In order that a space conformal to V_n admit G_r as a group of homothetic transformations with the same homothetic constants c_a, it is necessary and sufficient that there exist a function ρ such that $X_a(\rho^2 g_{\mu\nu}) = 2c_a \rho^2 g_{\mu\nu}$, from which $X_a \rho^2 = 0$. Now, if we assume that $r < n$, then these equations are completely integrable by virtue of $(X_b X_c)\rho^2 = c_{bc}{}^a X_a \rho^2$ and admit $n-q$ functionally independent solutions, q being the generic rank of the matrix $\| \xi_a^\lambda \|$. Thus we have

THEOREM 17. *If a Riemannian space V_n admits an r-parameter group G_r of homothetic transformations $(r < n)$, then there exist $n-q$ functionally independent spaces conformal to*

V_n and admitting G_r as group of homothetic transformations with the same homothetic constants, where q is the generic rank of $\| \xi_a^\lambda \|$.

This is a generalization of a theorem due to M. S. Knebelman[2].

We assume again that our space V_n admits an r-parameter group G_r of homothetic transformations, then we have $X_a g_{\mu\nu} = 2c_a g_{\mu\nu}$. In order that a space conformal to V_n admit G_r as group of motions, it is necessary and sufficient that there exist a function ρ^2 such that $X_a(\rho^2 g_{\mu\nu}) = 0$, from which $X_a \log \rho = -c_a$. In the case of $r \leqslant n$, these equations are completely integrable by virtue of $(X_b X_c) \log \rho = c_{bc}{}^a X_a \log \rho$ and $c_{bc}{}^a c_a = 0$. Thus we have

THEOREM 18. *If a Riemannian space V_n admits an r-parameter group G_r of homothetic transformations $(r \leqslant n)$, then there exists a space conformal to V_n which admits G_r as group of motions.*

6. Now we consider an r-parameter group G_r of general conformal transformations, then we have $X_a g_{\mu\nu} = 2\phi_a g_{\mu\nu}$, ϕ_a being r scalars. A necessary and sufficient condition that the group G_r contain a sub-group of homothetic transformations is that there exist constants c^a not all zero such that $c^a \phi_a =$ const. Successive covariant differentiation of this equation will give $c^a \phi_{a;\nu_1} = 0$, $c^a \phi_{a;\nu_1;\nu_2} = 0, \dots$ If we denote by $\alpha_1, \alpha_2, \dots$ the ranks of the matrices $\| \phi_{a;\nu_1} \|$, $\| \phi_{a;\nu_1}, \phi_{a;\nu_1;\nu_2} \|, \dots$ respectively, then we have $\alpha_1 \leqslant \alpha_2 \leqslant \dots$ But, since the above equations admit a set of solutions which are not all zero, we have $\alpha_1 \leqslant \alpha_2 \leqslant \dots < r$. But we can easily prove that, if $\alpha_\rho = \alpha_{\rho+1}$, then $\alpha_{\rho+1} = \alpha_{\rho+2}$. Thus the ranks of the matrices must satisfy

$$\alpha_1 \leqslant \alpha_2 \leqslant \dots \leqslant \alpha_\rho = \alpha_{\rho+2} = \dots = s < r. \qquad (6.1)$$

Conversely, if the above ranks $\alpha_1, \alpha_2, \dots,$ satisfy the relation (6.1), then we can find out a linearly indepen-

dent sets of solutions $f^a_A(x)$ $(A,B,C,\ldots = 1,2,\ldots,s)$ which are not all zero and such that

$$f^a_A\phi_{a;\nu_1}=0,\ f^a_A\phi_{a;\nu_1;\nu_2}=0,\ \ldots,\ f^a_A\phi_{a;\nu_1;\nu_2;\ldots;\nu_p}=0. \quad (6.2)$$

Differentiating these equations covariantly, and taking account of the fact that the ranks of matrices satisfy (6.1), we find that $f^a_{A;\nu}$ are also solutions of (6.2). Thus there must exist a set of functions $P^A{}_{B\nu}$ such that $f^a_{B;\nu}=f^a_A P^A{}_{B\nu}$ whose integrability conditions are

$$P^A{}_{B\nu;\omega}-P^A{}_{B\omega;\nu}+P^C{}_{B\nu}P^A{}_{C\omega}-P^C{}_{B\omega}P^A{}_{C\nu}=0. \quad (6.3)$$

But these equations show that there exists a set of functions $f^B(x)$ such that $f^B(x)f^a_B(x)=$ constant. In fact, the equations $(f^Bf^a_B)_{;\nu}=0$ give $f^A{}_{;\nu}+f^B P^A{}_{B\nu}=0$ which are completely integrable because of (6.3). Thus putting $c^a=f^Bf^a_B$, we find $c^a_{\cdot}\phi_{a;\nu}=0$ and consequently $c^a\phi_a=$ const. Thus we have

THEOREM 19. *A necessary and sufficient condition that an r-parameter group of conformal transformations whose defining equations are $X_a g_{\mu\nu}=2\phi_a g_{\mu\nu}$ contain a sub-group of homothetic transformations is that the ranks $\alpha_1, \alpha_2, \ldots$ of the matrices $\|\phi_{a;\nu_1}\|$, $\|\phi_{a;\nu_1}, \phi_{a;\nu_1;\nu_2}\|, \ldots$ satisfy the relation* (6.1).

Next, we consider an r-parameter group G_r of affine collineations, then we have $X_a\left\{ {\lambda \atop \mu\nu} \right\}=0$. A necessary and sufficient condition that the group G_r contain a sub-group of homothetic transformations is that there exist constants c^a which are not all zero, and c such that $c^aX_a g_{\mu\nu}=2cg_{\mu\nu}$. Thus the rank of the matrix $\|X_a g_{\mu\nu}, g_{\mu\nu}\|$ with $r+1$ rows and $\frac{1}{2}n(n+1)$ columns must be less than $r+1$.

Conversely if the rank s of the matrix $\|X_a g_{\mu\nu}, g_{\mu\nu}\|$ is less than $r+1$, then we can find out $r+1-s$ linearly independent solutions $g^a_L(x)$, $g_L(x)$ of $c^aX_a g_{\mu\nu}=2cg_{\mu\nu}$ such that

$$g^a_L(x)X_a g_{\mu\nu}=2g_L(x)g_{\mu\nu} \quad (L, M, N=1, 2, \ldots, r+1-s). \quad (6.4)$$

Covariant differentiation of these equations and $(X_a g_{\mu\nu})_{;\omega}=0$ which are equivalent to $X_a\left\{ {\lambda \atop \mu\nu} \right\}=0$ tell us

that $g^a_{L;\omega}$ and $g_{L;\omega}$ are also solutions of (6.4). Thus there must exist a set of functions such that

$$g^a_{M;\omega} = g^a_L Q^L_{M\omega},\ g_{M;\omega} = g_L Q^L_{M\omega}$$

whose integrability conditions are

$$Q^L_{M\omega;\sigma} - Q^L_{M\sigma;\omega} + Q^N_{M\omega} Q^L_{N\sigma} - Q^N_{M\sigma} Q^L_{N\omega} = 0. \qquad (6.5)$$

But these equations show that there exists a set of functions $g^L(x)$ such that $g^L(x)g^a_L(x) = \text{const.}$, $g^L(x)g_L(x) = \text{const.}$ Thus we have

THEOREM 20. *A necessary and sufficient condition that an r-parameter group of affine collineations contain a subgroup of homothetic transformations is that the rank of the matrix* $\| X_a g_{\mu\nu}, g_{\mu\nu} \|$ *be less than* $r+1$.

This is a generalization of a theorem due to M. S. Knebelman[3].

7. Next, we consider an r-parameter group G_r of transformations in an n-dimensional space ($r \leqslant n$) and suppose that there exist constants c_a not all zero such that $c_{bc}{}^a c_a = 0$, $c_{bc}{}^a$ being the constants of structure of the group G_r. Then our problem is the following: is it possible to introduce a Riemannian metric $g_{\mu\nu}$ with respect to which the given group G_r is a group of homothetic transformations with homothetic constants c_a? This problem will be reduced to the solution of a system of partial differential equations $X_a g_{\mu\nu} = 2c_a g_{\mu\nu}$ with unknowns $g_{\mu\nu}$.

We consider first the case in which the generic rank of the matrix $\| \xi^\lambda_a \|$ is equal to r. Then we can choose a coordinate system in which $| \xi^e_a | \neq 0$, $\xi^P_a = 0$ $(P = r+1, \ldots, n)$. If we define the inverse matrix $\| \eta^a_e \|$ of $\| \xi^e_a \|$, then we can solve the equations $X_a g_{\mu\nu} = 2c_a g_{\mu\nu}$ with respect to $\partial g_{\mu\nu}/\partial x^e$ obtaining

$$\frac{\partial g_{\mu\nu}}{\partial x^e} = H_{e\mu\nu}(x, g) + 2H_e(x) g_{\mu\nu}, \qquad (7.1)$$

where

$$H_{e\mu\nu} = -\eta^a_e \left(\frac{\partial \xi^f_a}{\partial x^\mu} g_{f\nu} + \frac{\partial \xi^f_a}{\partial x^\nu} g_{\mu f} \right), \tag{7.2}$$

$$H_e = \eta^a_e c_a. \tag{7.3}$$

We shall prove that the system (7.1) is completely integrable. By the definition of $H_{e\mu\nu}$, we have

$$X_c g_{\mu\nu} = \xi^f_c \left[\frac{\partial g_{\mu\nu}}{\partial x^f} - H_{f\mu\nu} \right],$$

$$X_b X_c g_{\mu\nu} = \xi^e_b \frac{\partial \xi^f_c}{\partial x^e} \left[\frac{\partial g_{\mu\nu}}{\partial x^f} - H_{f\mu\nu} \right]$$

$$+ \xi^e_b \xi^f_c \left[\frac{\partial^2 g_{\mu\nu}}{\partial x^e \partial x^f} - \frac{\partial H_{f\mu\nu}}{\partial x^e} - \frac{\partial H_{f\mu\nu}}{\partial g_{\beta\gamma}} \frac{\partial g_{\beta\gamma}}{\partial x^e} \right] + \frac{\partial \xi^e_b}{\partial x^\mu} X_c g_{e\nu} + \frac{\partial \xi^e_b}{\partial x^\nu} X_c g_{\mu e},$$

$\partial/\partial g_{\beta\gamma}$ being applied formally. Substituting these equations into $(X_b X_c) g_{\mu\nu} = c_{bc}{}^a X_a g_{\mu\nu}$, and taking account of $(X_b X_c) f = c_{bc}{}^a X_a f$, of $X_a g_{\mu\nu} = 2 c_a g_{\mu\nu}$ and of $|\xi^e_b| \neq 0$, we find

$$\frac{\partial H_{e\mu\nu}}{\partial x^f} + \frac{\partial H_{e\mu\nu}}{\partial g_{\beta\gamma}} (H_{f\beta\gamma} + 2 H_f g_{\beta\gamma}) + 2 H_e H_{f\mu\nu}$$

$$= \frac{\partial H_{f\mu\nu}}{\partial x^e} + \frac{\partial H_{f\mu\nu}}{\partial g_{\beta\gamma}} (H_{e\beta\gamma} + 2 H_e g_{\beta\gamma}) + 2 H_f H_{e\mu\nu}. \tag{7.4}$$

On the other hand, by the definition of H_e, we have

$$X_c g_{\mu\nu} = 2 \xi^f_c H_f g_{\mu\nu},$$

$$X_b X_c g_{\mu\nu} = 2 \xi^e_b \frac{\partial \xi^f_c}{\partial x^e} H_f g_{\mu\nu} + 2 \xi^e_b \xi^f_c \frac{\partial H_f}{\partial x^e} g_{\mu\nu} + 2 \xi^e_b \xi^f_c H_f \frac{\partial g_{\mu\nu}}{\partial x^e}$$

$$+ \frac{\partial \xi^e_b}{\partial x^\mu} X_c g_{e\nu} + \frac{\partial \xi^e_b}{\partial x^\nu} X_c g_{\mu e}.$$

Substituting these equations into $(X_b X_c) g_{\mu\nu} = c_{bc}{}^a X_a g_{\mu\nu}$ and taking account of $(X_b X_c) f = c_{bc}{}^a X_a f$, of $X_a g_{\mu\nu} = 2 c_a g_{\mu\nu}$, of $|\xi^e_b| \neq 0$ and of $c_{bc}{}^a c_a = 0$, we find

$$\frac{\partial H_e}{\partial x^f} = \frac{\partial H_f}{\partial x^e}. \tag{7.5}$$

From (7.4) and (7.5), we get

$$\frac{\partial H_{e\mu\nu}}{\partial x^f} + \frac{\partial H_{e\mu\nu}}{\partial g_{\beta\gamma}} (H_{f\beta\gamma} + 2 H_f g_{\beta\gamma}) + 2 \frac{\partial H_e}{\partial x^f} g_{\mu\nu}$$

$$+ 2 H_e (H_{f\mu\nu} + 2 H_f g_{\mu\nu})$$

$$= \frac{\partial H_{f\mu\nu}}{\partial x^e} + \frac{\partial H_{f\mu\nu}}{\partial g_{\beta\gamma}} (H_{e\beta\gamma} + 2\, H_e g_{\beta\gamma}) + 2\, \frac{\partial H_f}{\partial x^e}\, g_{\mu\nu}$$
$$+ 2\, H_f\, (H_{e\mu\nu} + 2\, H_e\, g_{\mu\nu}), \qquad (7.6)$$

which shows that the system (7.1) is completely integrable. Thus we have

THEOREM 21. *An r-parameter group of transformations in an n-dimensional space $(r \leqslant n)$ such that the generic rank of the matrix $\| \xi^\lambda_a \|$ is r and there exist r constants c_a not all zero such that $c_{bc}{}^a\, c_a = 0$, can be a group of homothetic transformations with homothetic constants c_a of a Riemannian space whose fundamental tensor involves $\frac{1}{2} n\, (n+1)$ arbitrary functions.*

We next consider the case in which the generic rank q of the matrix $\| \xi^\lambda_a \|$ is less than r $(\leqslant n)$ and assume that $| \xi^i_j | \neq 0$, $\xi^\lambda_l = \varphi^i_l \xi^\lambda_i$ $(h, i, j, k = 1, 2, \ldots q;\ l, m, n = q+1, \ldots, r)$. Then we can choose a coordinate system in which $| \xi^i_j | \neq 0$, $\xi^P_j = 0$ $(P = q+1, \ldots, n)$. If we define the inverse matrix $\| \eta^i_k \|$ of $\| \xi^i_j \|$, then the equations $X_a\, g_{\mu\nu} = 2\, c_a\, g_{\mu\nu}$ give

$$\frac{\partial g_{\mu\nu}}{\partial x^j} = H_{j\mu\nu}\, (x, g) + 2\, H_j\, (x)\, g_{\mu\nu}, \qquad (7.7)$$

$$A_{l\mu\nu}\, (x, g) + 2\, (\varphi^i_l\, c_i - c_e)\, g_{\mu\nu} = 0, \qquad (7.8)$$

where

$$H_{j\mu\nu}\, (x, g) = -\eta^i_j \left(\frac{\partial \xi^k_i}{\partial x^\mu}\, g_{k\nu} + \frac{\partial \xi^k_i}{\partial x^\nu}\, g_{\mu k} \right), \qquad (7.9)$$

$$H_j\, (x) = \eta^i_j\, c_i, \qquad (7.10)$$

$$A_{l\mu\nu}\, (x, g) = \left(\frac{\partial \varphi^j_l}{\partial x^\mu} g_{i\nu} + \frac{\partial \varphi^j_l}{\partial x^\nu}\, g_{\mu i} \right) \xi^i_j. \qquad (7.11)$$

By a similar method as above, we can prove that

$$\frac{\partial H_{i\mu\nu}}{\partial x^h} + \frac{\partial H_{i\mu\nu}}{\partial g_{\beta\gamma}}\, (H_{h\beta\gamma} + 2\, H_h\, g_{\beta\gamma}) + 2\, \frac{\partial H_i}{\partial x^h}\, g_{\mu\nu}$$
$$+ 2\, H_i\, (H_{h\mu\nu} + 2\, H_h\, g_{\mu\nu})$$
$$= \frac{\partial H_{h\mu\nu}}{\partial x^i} + \frac{\partial H_{h\mu\nu}}{\partial g_{\beta\gamma}}\, (H_{i\beta\gamma} + 2\, H_i\, g_{\beta\gamma}) + 2\, \frac{\partial H_h}{\partial x^i}\, g_{\mu\nu}$$
$$+ 2\, H_h\, (H_{i\mu\nu} + 2\, H_i\, g_{\mu\nu}), \qquad (7.12)$$

$$\frac{\partial A_{l\mu\nu}}{\partial x^i}+\frac{\partial A_{l\mu\nu}}{\partial g_{\beta\gamma}}(H_{i\beta\gamma}+2\,H_i\,g_{\beta\gamma})+2\,\frac{\partial(\varphi_l^k\,c_k-c_e)}{\partial x^i}\,g_{\mu\nu}$$
$$+2\,(\varphi_l^k\,c_k-c_l)\,(H_{i\mu\nu}+2\,H_i\,g_{\mu\nu})=0. \quad (7.13)$$

Equations (7.12) and (7.13) show that the mixed system (7.7) and (7.8) is completely integrable. Thus we have

THEOREM 22. *The generic rank q of the matrix $\|\xi_a^\lambda\|$ being less than r for an r-parameter group of transformations in an n-dimensional space $(r \leqslant n)$, if in the coordinate system in which $|\xi_j^i| \neq 0$, $\xi_i^Q=0$ $(i, j=1, \ldots, q;\ Q=q+1, \ldots, n)$, the equations $c_{bc}{}^a\,c_a=0$ and $A_{l\mu\nu}(x, g)+2\,(\varphi_l^i\,c_i-c_l)\,g_{\mu\nu}=0$ are consistent in the c_a and $g_{\mu\nu}$ $(|g_{\mu\nu}|\neq 0)$ when the coordinates are given particular values, then there exists a Riemannian metric for which the given group is the group of homothetic transformations with homothetic constants c_a.*

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THE CONVOLUTIONS OF THE KERNEL $x^\lambda - t^\lambda$

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Introduction. Convolutions of functions of two variables were introduced by Volterra as a means of solving certain types of integral equations. The method which he and his followers used to obtain representations of convolutions of arbitrary order is formalistic, and hardly practical, except for kernels of the type $h(x-t)$, the so-called "closed cycle". A different method has been developed recently by D. G. Bourgin and employed by him (D. G. Bourgin: Associated transforms and convolutions, *Revista de Ciencias*, Lima, 51 (1949), 5-46), to obtain representations for the convolutions of the kernel $x+t$. His method, which depends on an associated transform (and is, in effect, a generalization of the Laplace transform theory) is employed here to treat the kernel $x^\lambda - t^\lambda$. Representations for these convolutions are also obtained by yet another method prompted by the work of Erdélyi. It rests on the concepts of fractional integration and differentiation.

1. **Outline of the transform method.** The kernel to be discussed is defined by

$$\begin{aligned} h(x, t) &= x^\lambda - t^\lambda, \quad & x \geqslant t \geqslant 0, \\ &= 0, & x < t,\ t < 0, \end{aligned}$$

and $\lambda > 0$. The positive integral powers of convolution are given by

$$h^n(x,y) = \int_y^x h^{n-1}(x, t)\, h(t, y)dt, \quad n = 2, 3, 4, \ldots \qquad (1)$$

The non-integral powers are those functions $h^\gamma(x, y)$, $\gamma > 0$, which reduce to the corresponding integral power when γ is an integer and which in addition satisfy the relation

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$$h^{\alpha+\beta}(x,y) = \int_y^x h^\alpha(x,t)h^\beta(t,y)\,dt, \quad \alpha, \beta > 0. \qquad (2)$$

The transform method depends upon a consideration of

$$p^{-1}f(t) = \int_0^x h(x,t)f(t)\,dt,$$

where p^{-1} is regarded as a function of the complex variable p, the map of the kernel $h(x, t)$ determined by the operator, rather than as an integration operator. The associated transform kernel $L(p, x)$ then satisfies

$$\int_0^\infty L(p,x)\int_0^x h(x,t)f(t)\,dt\,dx = \int_0^\infty p^{-1}L(p,x)f(x)\,dx,$$

which for a suitable set of functions $f(t)$ implies

$$L(p,x)p^{-1} = \int_x^\infty L(p,t)\,h(t,x)\,dt.$$

By an induction argument, it follows that the integral powers of convolution of $h(x, t)$ are given by

$$h^n(t,x) = \mathbf{S}L(p,x)\,p^{-n},$$

where $\mathbf{S}$ is the inverse operator to

$$\int_0^\infty L(p,t).\ (\quad).\,dt.$$

2. **The transform method for the kernel** $x^\lambda - t^\lambda$. The associated transform kernel $L(p, x)$ is defined as a solution of the integral equation

$$L(p,x)p^{-1} = \int_x^\infty L(p,t)\,(t^\lambda - x^\lambda)\,dt. \qquad (3)$$

As suggested by Bourgin, it is possible to solve this equation by throwing it over into a partial differential equation, or rather a total differential equation in x with p regarded as a parameter. Upon differentiation of (3) with respect to x, isolation of the integral, and differentiation again there is obtained

$$x^2\frac{d^2L}{dx^2} + (1-\lambda)\,x\frac{dL}{dx} - \lambda\,px^{\lambda+1}L = 0, \qquad (4)$$

where $L(p, x)$ and $x^\lambda L(p, x)$ are to be integrable on $0 \leqslant x < \infty$ and $L(p, x)$ together with its first two deriva-

tives are to be continuous for the same range, all for some domain of p. This equation is of the Bessel type and the solution satisfying the initial conditions is most conveniently expressed in terms of the Macdonald Bessel function of imaginary argument, $K_\nu(z)$, [10, pp. 78 and 97]:

$$L(p, x) = (qu)^{\nu/2} K_\nu(2\sqrt{qu}), \tag{5}$$

where (here and in the sequel) it is convenient to employ the notation

$$\begin{cases} \nu = \dfrac{\lambda}{\lambda+1}, \\ q = \dfrac{\lambda p}{(\lambda+1)^2}, \end{cases} \qquad \begin{cases} u = x^{\lambda+1}, \\ v = t^{\lambda+1}, \\ z = y^{\lambda+1}. \end{cases}$$

The domain of p used here and throughout is defined by $|\arg p| < \pi$. That (5) which satisfies the differential equation (4) derived from the integral equation (3) does actually satisfy the latter may be checked in various ways. We can, for example, employ an integral representation of the Barnes type for the $K_\nu(z)$ function [10, p. 192] in the right side of (3) and interchange the order of integration (a process easily justified on the basis of absolute integrability), carry out the inner elementary integration, and then identify the resulting integral as an integral representation of the same type for the left member of (3). Or, alternatively, we may take the Mellin transform of (3) and apply Kober's fractional integration theory [7, Th. 5] together with the known Mellin transform of the $K_\nu(z)$ function [10, p. 388].

The inverse operator **S** for our case is to be found in the work of Meijer [9] (cf. also [1], [2]) and in our notation is

$$\mathbf{S}f(q) = \frac{v^{\nu/2}}{(1-\nu)\pi i} \int_{-\infty}^{(0+)} I_\nu(2\sqrt{(qv)})\, q^{-\nu/2} f(q)\, dq, \tag{6}$$

where the limits on the integral are to imply that the path of integration is a Hankel loop starting at $-\infty$ with

$\arg q = -\pi$, swinging around the origin in the positive direction, and then returning to $-\infty$ with $\arg q = +\pi$. $I_\nu(z)$ is the Bessel function of the first kind with imaginary argument [10, p. 77].

It may be remarked here, as being of interest in the transform theory, that the inversion formula just given is not unique, at least for the operators applied to the power function. There is for such functions a more general kernel for **S** which includes that of (6) as a special case. It can be derived by first finding the value [10, p. 388] of

$$g(p) = \int_0^\infty L(p, t)\, t^\mu dt, \quad \mathcal{R}(\mu) > -1,$$

and then seeking a kernel $W(p, t)$ so that

$$\frac{1}{2\pi i}\int_{\odot} W(p, t) g(p)\, dp = t^\mu. \tag{7}$$

This can be accomplished by first restricting μ to be of the form $\mu = (n+\alpha)(\lambda+1)-1$, $n = 1, 2, 3, \ldots$ and α a real number lying in an interval determined later and then assuming a power series expansion for $W(p, t)$ of the form

$$W(p, t) = \sum_{s=0}^{\infty} a_s t^{(\lambda+1)(s+\alpha)+\lambda} p^{s+\alpha}.$$

In this case the theory of residues may be applied to (7) to determine the a_s. The restriction on the form of μ is then lifted by replacing the circular contour by a Hankel loop (such as that employed in (6)), and the integrand is made single valued by introducing a cut along the negative axis of reals. The kernel $W(p, t)$ is of the generalized hypergeometric type and the final inversion formula can be written as

$$\frac{1}{\pi i \Gamma(1+\alpha)\Gamma(1+\alpha+\nu)} \int_{-\infty}^{(0+)} {}_1F_2(1;\ 1+\alpha,\ 1+\alpha+\nu;\ qv)(qv)^\alpha \int^\infty K_\nu(2\sqrt{(qu)})(qu)^{\nu/2} u^\sigma du\, dq = v^\sigma \tag{8}$$

wherein we have set $(1-\nu)\mu-\nu=\sigma$. This formula can be checked by employing an integral representation for the generalized hypergeometric function [10, pp. 346, 374] and interchanging the order of integration. In order to insure the validity of these computations certain restrictions must be placed on α and μ but by analytic continuation these may be lightened so that (8) holds for $-\nu<\alpha<1-\nu$, $0\leqslant\nu<1$, and $\mathcal{R}(\mu)\geqslant 0$. When $\alpha=0$ the inversion operator in (8) reduces to (6), which is the one used in the sequel. Formula (8) is noteworthy in that the inversion kernel $W(p, t)$ admits of an arbitrary continuous parameter, α.

Using the inversion operator (6) we can write down immediately an integral representation for the integral powers of convolution. It is

$$h^n(x,t)=\frac{\nu^n(1-\nu)^{n-1}(uv)^{\nu/2}}{\pi i}\int_{-\infty}^{(0+)}I_\nu(2\sqrt{(qu)})K_\nu(2\sqrt{(qv)})q^{-n}dq.$$

At $q=-\infty$ the integrand is $O(|q^{-n-\frac{1}{2}}|)$, where the constant implied in the O symbol is independent of n. Consequently, the integral converges and represents an analytic function of n for $\mathcal{R}(n)>\frac{1}{2}$ and thus gives the integral powers of convolution ($n=1, 2, 3, \ldots$) of the kernel $x^\lambda-t^\lambda$.

This integral has a rather simple representation in terms of the hypergeometric function. To obtain it, deform the contour to two lines along the real axis plus a small circle around the origin, and, to make use of known integrals, express the Bessel functions $I_\nu(z)$ and $K_\nu(z)$ in terms of the Bessel function $J_\nu(z)$ [10, p. 77]. It is easy to show from the known bound on $J_\nu(z)$ [10, p. 49) that the integral around the small circle vanishes with the radius of the circle provided that $\frac{1}{2}<\mathcal{R}(n)<1$. Imposing this restriction on n and letting the

radius vanish we find that, with an obvious transformation,

$$h^\nu(x, t) = \frac{2\nu^n(1-\nu)^{n-1}(uv)^{\nu/2}}{\sin(\nu\pi)}$$

$$\times \left\{ \sin n\pi \int_0^\infty J_\nu(2\omega\sqrt{u}) J_{-\nu}(2\omega\sqrt{v})\,\omega^{-2n+1} d\omega \right.$$
$$\left. - \sin(n-\nu)\pi \int_0^\infty J_\nu(2\omega\sqrt{u}) J_\nu(2\omega\sqrt{v})\,\omega^{-2n+1} d\omega \right\}. \qquad (9)$$

The values of the integrals are known [10, p. 401], and in the case $0 < t < x$ (i.e. $0 < v < u$) they give

$$h^n(x, t) = \frac{\pi\nu^n(1-\nu)^{n-1} u^{(n-\frac{1}{2})\nu} v^{\nu/2}}{\Gamma(n)\sin\nu\pi}$$

$$\times \left\{ \frac{(uv^{-1})^{\nu/2}}{\Gamma(1-\nu)\Gamma(n+\nu)}\, {}_2F_1(-n+1,\ -n+1-\nu;\ 1-\nu;\ u^{-1}v) \right.$$
$$\left. - \frac{(u^{-1}v)^{\nu/2}}{\Gamma(1+\nu)\Gamma(n-\nu)}\, {}_2F_1(-n+1,\ -n+1+\nu;\ 1+\nu;\ u^{-1}v) \right\}. \qquad (10)$$

This result, established for $\frac{1}{2} < \mathcal{R}(n) < 1$, may be extended to $\mathcal{R}(n) > \frac{1}{2}$ by the theory of analytic continuation, so that (10) gives all the integral powers of convolution of $x^\lambda - t^\lambda$.

If $0 < x \leqslant t$ (i. e. $0 < u \leqslant v$) then the evaluation as above for the integral gives zero as the powers of convolution. This is in accordance with the general theory since $h(x, t) = 0$ for $0 < x \leqslant t$.

It is possible to obtain a compact representation involving only one hypergeometric function for the convolutions by employing one of Kummer's identities [8, p. 39], whence

$$h^n(x, t) = \frac{\nu^n(1-\nu)^{n-1}}{\Gamma(2n)} u^{n+\nu-1}(1-u^{-1}v)^{2n-1}$$
$$\times\, {}_2F_1(n-\nu,\ n;\ 2n;\ 1-u^{-1}v). \qquad (11)$$

Note that this new hypergeometric function is an infinite series, not a polynomial as in (10), and that it converges for $x \geqslant t > 0$, as desired.

We now show that this final expression also gives the non-integral powers of convolution when n, no longer restricted to be an integer, is replaced by α where $\alpha > 0$. That is, $h^{\alpha}(x,y)$ satisfies (2). This functional equation to be verified is, in our case

$$
{}_2F_1(\alpha+\beta-\nu,\ \alpha+\beta;\ 2\alpha+2\beta;\ \xi) = \frac{\Gamma(2\alpha+2\beta)}{\Gamma(2\alpha)\Gamma(2\beta)}
$$

$$
\times\int_0^1 s^{2\alpha-1}(1-s)^{2\beta-1}(1-\xi s)^{-\beta}{}_2F_1(\alpha-\nu;\ \alpha;\ 2\alpha;\ \xi s)
$$

$$
\times\,{}^2F_1\left(\beta-\nu,\ \beta;\ 2\beta;\ \frac{\xi(1-s)}{1-\xi s}\right)ds \qquad (12)
$$

in which there have been set

$$
\begin{cases} s = \dfrac{u-v}{u-z}, \\ \xi = 1-u^{-1}z. \end{cases}
$$

To verify that (12) is true we need but note that it is a special case of a formula due to Erdélyi [3, p. 182] obtained by replacing a, b, c, l, m of his formula with $\alpha+\beta$, $\alpha+\beta-\nu$, $2\alpha+2\beta$, $2\alpha+\beta-\nu$, and 2α, respectively. His more general formula he showed to hold under the conditions $\mathcal{R}(c) > \mathcal{R}(m) > 0$, $\xi \neq 1$, $|\arg(1-\xi)| < \pi$. The restriction $\mathcal{R}(c) > \mathcal{R}(m) > 0$ happens to be a natural one in our theory since it becomes $\mathcal{R}(\alpha) > 0$, $\mathcal{R}(\beta) > 0$. Taking our cue from this we may speak of $h^{\alpha}(x, y)$ as giving the "complex powers of convolution" of the kernel $x^{\lambda}-y^{\lambda}$. The condition $|\arg(1-\xi)| < \pi$ is immediately satisfied since $1-\xi = (x^{-1}y)^{\lambda+1}$ is real and positive. The restriction $\xi \neq 1$ would seem to imply that the case $y = 0$ must be excluded. However, although this restriction on ξ is necessary in Erdélyi's general formula, it may be removed in our special case (12). We show this by justifying the passage to the limit, $\xi \to 1$, under the integral sign in (12). For any value of s between 0 and 1 the functions $(1-\xi s)^{-\beta}$, ${}_2F_1(\alpha-\nu,\ \alpha;\ 2\alpha;\ \xi s)$ and ${}_2F_1(\beta-\nu,\ \beta;\ 2\beta;\ \xi(1-s)/(1-\xi s)$ are continuous in ξ for $0 \leqslant \xi \leqslant 1$ (the last function by Abel's Theorem) and therefore are uniformly continuous there. Thus it follows that the

limit as $\xi \to 1$ is uniform in s for any closed interval interior to $(0, 1)$. Also it is evident that the integrand is continuous in s over such an interval for each $\xi < 1$. Although in general the hypergeometric function is not finite for unit argument, those appearing in the integrand of (12) do happen to be [11, p. 25]. Consequently, the integrand is dominated in the open interval $0 < s < 1$ by the integrable function $Cs^{2\alpha-1}(1-s)^{\beta-1}$ for $\xi < 1$, since $(1-\xi s)^{-\beta} < (1-s)^{-\beta}$. The passage to the limit under the integral sign is therefore permissible[6].

Inasmuch as (12) holds for more general λ, even complex values, the representation (11) need not be restricted by λ real and positive. In fact, λ may be any complex number except -1 provided that $|\log(x/y).\mathscr{I}(\lambda+1)| < \pi$ (as this is what the condition $|\arg(1-\xi)| < \pi$ becomes). The condition $\xi \neq 1$ must be imposed, however, unless $\mathscr{R}(\lambda/(\lambda+1)) > 0$, in which case it may be removed as for $\lambda > 0$. The exceptional case $\lambda = -1$ may be treated by a limiting process as in the preceding paragraph. We note that

$$\lim_{\lambda \to -1} \frac{1-(x^{-1}y)^{\lambda+1}}{\lambda+1} = \log\frac{x}{y}$$

and then use the method that Kummer used to establish the formula [8, p. 140]

$$\lim_{\varepsilon \to 0} {}_2F_1(\alpha, \beta/\epsilon; \gamma; \epsilon z) = {}_1F_1(\alpha; \gamma; z)$$

in order to get the representation given below for this exceptional case.

Summarizing the results, then, gives for the convolutions in the original variables

$$h^\alpha(x, y) = \frac{\lambda^\alpha x^{\alpha(\lambda+1)-1}}{(\lambda+1)^{2\alpha-1}\Gamma(2\alpha)} (1-(x^{-1}y)^{\lambda+1})^{2\alpha-1}$$

$$\times\ {}_2F_1(\alpha - \frac{\lambda}{\lambda+1}, \alpha;\ 2\alpha;\ 1-(x^{-1}y)^{\lambda+1}),\ \lambda \neq -1$$

$$= \frac{e^{\pi i\alpha}}{\Gamma(2\alpha)} x^{-1} (\log\frac{x}{y})^{2\alpha-1}\ {}_1F_1(\alpha;\ 2\alpha;\ \log\frac{x}{y}),\ \lambda = -1,$$

where $\mathscr{R}(\alpha) > 0$, $|\log\ (x/y) . \mathscr{I}(\lambda+1)| < \pi$, and, if $\mathscr{R}\left(\frac{\lambda}{\lambda+1}\right) \leqslant 0$, $(x^{-1}y)^{\lambda+1} \neq 0$.

3. **The convolutions by fractional integration by parts.** The method that Erdélyi used to establish his more general formulation of (12) is singularly well adapted towards deriving representations for the convolutions of the kernel $x^\lambda - t^\lambda$ and perhaps for others as well. This alternative scheme to the transform method rests on the concepts of fractional differentiation and integration. Fractional integration as employed by Erdélyi [3] is defined in the following manner:

$$\frac{d^{-\sigma} f(x)}{d(x-a)^{-\sigma}} = \frac{1}{\Gamma(\sigma)} \int_a^x (x-y)^{\sigma-1} f(y)\, dy,$$

$$\mathscr{R}(\sigma) > 0. \quad (13)$$

$$\frac{d^{-\sigma} f(x)}{d(b-x)^{-\sigma}} = \frac{1}{\Gamma(\sigma)} \int_x^b (y-x)^{\sigma-1} f(y)\, dy.$$

Fractional differentiation, on the other hand, is defined with reference to a particular function, x^μ:

$$\frac{d^\sigma x^\mu}{dx^\sigma} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\sigma+1)} x^{\mu-\sigma}, \quad \mu-\sigma+1 \neq 0. \quad (14)$$

It follows that for the function x^μ these definitions in the common range of validity coincide, and (14) will frequently be found convenient to use for fractional integration as well as fractional differentiation. Fractional integration by parts is expressed in the compact form

$$\int_a^b \frac{d^\sigma f}{d(x-a)^\sigma} g\, dx = \int_a^b f \frac{d^\sigma g}{d(b-x)^\sigma}\, dx. \quad (15)$$

The proposed method for obtaining convolution representations is to start with (1) for $n = 2$ and transform it by (15). Do the same for one or two further values of n and from all these infer the general integral convolutions. As in the transform method it is convenient at first in our case to assume $\lambda > 0$. We have

$$h^2(x, y) = (1-\nu)\, u^{1+\nu} \int_0^\xi [1-(1-s)^\nu]\left[1-\left(\frac{1-s}{1-\xi}\right)^{-\nu}\right] ds, \quad (16)$$

by employing the substitutions

$$\begin{cases} s = 1 - u^{-1}v, \\ \xi = 1 - u^{-1}z. \end{cases}$$

Let the first and second bracketed expressions in (16) be the f and g, respectively, of (15) and choose $\sigma = 1 - \nu$. Expanding in a power series in s and applying (14) the first gives

$$\frac{d^{\nu-1}}{ds^{\nu-1}}\left[1-(1-s)^{\nu}\right] = -\frac{\Gamma(1-\nu)}{\Gamma(-\nu)\,\Gamma(3-\nu)}\,{}_2F_1\,(1, 1-\nu; 3-\nu; s)\, s^{2-\nu};$$

and, similarly, expanding the second in powers of $\xi - s$ results in

$$\frac{d^{1-\nu}}{d(\xi-s)^{1-\nu}}\left[1-\left(1-\frac{\xi-s}{\xi-1}\right)^{-\nu}\right] = \frac{1}{\Gamma(\nu)}\,(\xi-s)^{\nu}\,(1-s)^{-1}.$$

Hence it follows from (15) and an identity due to Euler [11, p. 286], whereby the factor $(1-s)^{-1}$ can be absorbed in the hypergeometric function, that

$$h^2\,(x, y) = -\frac{\Gamma(2-\nu)\,u^{1+\nu}}{\Gamma(-\nu)\,\Gamma(\nu)\,\Gamma(3-\nu)}$$

$$\times \int_0^{\xi} {}_2F_1\,(2-\nu, 2;\; 3-\nu;\, s)\, s^{2-\nu}\,(\xi-s)^{\nu}\, ds.$$

But this integral is of the form of the first of (13) where $\sigma = \nu + 1$ and can be evaluated by the alternative (14). The result is

$$h^2\,(x, y) = \frac{\nu^2\,(1-\nu)\,u^{1+\nu}}{\Gamma(4)}\,\xi^3\,{}_2F_1\,(2-\nu, 2\,;\,4\,;\,\xi),$$

which is recognized as (11) for $n = 2$. A similar result is obtained using $n = 3$ in (1). On the basis of these two convolutions one might infer that (11) gives the general integral convolutions. This can be confirmed by induction following the work of the preceding section.

It may be noted here that Fox [5] and more particularly Erdélyi [4] have investigated a family of hypergeometric transforms whose kernels resemble the convolutions we have obtained. Erdélyi used both fractional integration by parts and Hankel transforms.

In the latter formulation his final formula is similar in appearance to (9) but differs from it in that in (9) the parameter *n* is on the power function in the integrand whereas in his, it is on the order of one of the Bessel functions.

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ON THE RELATIVE EXTREMA OF THE HERMITE ORTHOGONAL FUNCTIONS

By Otto Szasz

1. The Hermite polynomials $H_n(x)$ satisfy the recurrence relation

$$H_{n+1}(x) = 2x\,H_n(x) - 2n\,H_{n-1}(x),\ n \geqslant 1, \tag{1.1}$$

$$H_0(x) = 1,\ H_1(x) = 2x,\ H_2(x) = 4x^2 - 2,\ H_3(x) = 8x^3 - 12x. \tag{1.2}$$

We follow the notation of G. Szegö [3, pp. 101-102]. We have

$$z'' + (2n+1-x^2)\,z = 0,\ z = e^{-x^2/2}\,H_n(x) = z_n(x), \tag{1.3}$$

and

$$\int_{-\infty}^{\infty} z_n(x)\,z_m(x)\,dx = \pi^{\frac{1}{2}}\,2^n n!\ \delta_{n,m};\ n, m = 0, 1, 2, \ldots$$

$$\delta_{n,m} = \begin{cases} 0, & n \neq m \\ 1, & n = m. \end{cases}$$

We write

$$\phi_n(x) = \frac{z_n(x)}{(2^n n!)^{\frac{1}{2}}} = e^{-x^2/2}\,\frac{H_n(x)}{(2^n n!)^{\frac{1}{2}}},\ n = 0, 1, 2, \ldots. \tag{1.4}$$

We denote the successive relative maxima of $|\phi_n(x)|$, as x decreases from $+\infty$ to 0, by

$$\mu_{0,n},\ \mu_{1,n},\ \mu_{2,n}, \ldots. \tag{1.5}$$

It is known [3, p. 172] that the sequence (1.5) is decreasing

$$\mu_{r,n} > \mu_{r+1,n};\ r = 0, 1, \ldots.$$

Following a conjecture of John Todd for the classical orthogonal polynomials, we shall show that

$$\mu_{r,n} > \mu_{r,n+1},\ n \geqslant r \geqslant 0. \tag{1.6}$$

The analogous result for Legendre polynomials was proved by Szegö [2], for ultraspherical polynomials

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by the author [1], and for the Laguerre polynomials by Todd [4].

We shall use the formulae

$$H'_n(x) = 2nH_{n-1}(x) \tag{1.7}$$
$$H_n(x) = 2xH_{n-1}(x) - H'_{n-1}(x),$$

so that

$$H'_n(x) = 2xH_n(x) - H_{n+1}(x). \tag{1.8}$$

From (1.4) and (1.7)

$$H_n(x) = (2^n n!)^{\frac{1}{2}} e^{x^2/2} \phi_n(x), \tag{1.9}$$

and

$$H'_n = (2^n n!)^{\frac{1}{2}} e^{x^2/2} (\phi'_n + x\phi_n) = 2n(2^{n-1}(n-1)!)^{\frac{1}{2}} e^{x^2/2} \phi_{n-1},$$

so that

$$\phi'_n + x\phi_n = \sqrt{(2n)}\,\phi_{n-1}, \quad \phi'_n = \sqrt{(2n)}\,\phi_{n-1} - x\phi_n. \tag{1.10}$$

From (1.8) and (1.9)

$$(2^n n!)^{\frac{1}{2}} e^{x^2/2} (\phi'_n + x\phi_n) = (2^n n!)^{\frac{1}{2}} e^{x^2/2} 2x\phi_n$$
$$- (2^{n+1}(n+1)\,!)^{\frac{1}{2}} e^{x^2/2} \phi_{n+1},$$

so that

$$\phi'_n + x\phi_n = 2x\phi_n - \sqrt{2(n+1)}\,\phi_{n+1},$$
$$\phi_n' = x\phi_n - \sqrt{2(n+1)}\,\phi_{n+1}, \; n \geqslant 0. \tag{1.11}$$

From (1.10), replacing n by $n+1$

$$\phi'_{n+1} = \sqrt{2(n+1)}\,\phi_n - x\phi_{n+1}; \tag{1.12}$$

now from (1.11) and (1.12)

$$\phi'_n + \phi'_{n+1} = (x + \sqrt{2(n+1)})(\phi_n - \phi_{n+1}). \tag{1.13}$$

From (1.1) and (1.2)

$$H_n(-x) = (-1)^n H_n(x),$$

so that from (1.4)

$$\phi_n(-x) = (-1)^n \phi_n(x),$$

and now from (1.10)

$$\phi'_n(-x) = (-1)^{n-1} \phi'_n(x).$$

Replacing x by $-x$ in (1.13) we now get

$$\phi'_n(x) - \phi'_{n+1}(x) = -(\sqrt{2(n+1)} - x)\;(\phi_{n+1}(x) + \phi_n(x)) \tag{1.14}$$

$$= (x - \sqrt{2(n+1)})\ (\phi_n + \phi_{n+1}).$$

Multiplying (1.13) by (1.14) we obtain

$$\phi'^2_n - \phi'^2_{n+1} = (x^2 - 2(n+1))\ (\phi_n^2 - \phi^2_{n+1}),$$

so that

$$\phi^2_{n+1} - \phi^2_n = \frac{1}{x^2 - 2(n+1)} (\phi'^2_{n+1} - \phi'^2_n). \tag{1.15}$$

2. Consider the zeros of $\phi'_n(x)$, where (by (1.4))

$\phi'_n(x) = \frac{e^{-x^2/2}}{(2^n n!)^{\frac{1}{2}}} (H'_n x - H_n)$. Let

$$\eta_n(x) \equiv H'_n(x) - xH_n(x) = 0, \tag{2.1}$$

or by (1.8) consider the equation

$$\eta_n(x) \equiv xH_n(x) - H_{n+1}(x) = 0. \tag{2.2}$$

The zeros of ϕ_n are (by (1.4)) the n zeros of $H_n(x)$; by Rolle's theorem between any two roots of $\phi = 0$ lies at least one root of (2.1); (2.1) has $n+1$ roots.

It is known that $H_n(x)$ has all its zeros real and simple:

$$x_{1,n} > x_{2,n} > \cdots > x_{n-1,n} > x_{n,n}. \tag{2.3}$$

By (1.4) $\phi_n(x)$ vanishes at these points and at $x = \pm\infty$. By (2.1) ϕ'_n has $n+1$ roots; denoting them by

$$x'_{0,n} > x'_{1,n} > \cdots > x'_{n,n}, \tag{2.4}$$

it follows that

$$x'_{0,n} > x_{1,n} > x'_{1,n} > x_{2,n} > \cdots > x'_{n-2,n} > x_{n-1,n} > x'_{n-1,n} > x_{n,n} > x'_{n,n}, \tag{2.5}$$

and similarly,

$$x'_{o,n+1} > x_{1,n+1} > x'_{1,n+1} > x_{2,n+1} > \cdots.$$

The roots of H_n are by (1.7) the roots of H'_{n+1}, and are interlaced with the roots of H_{n+1}, thus

$$x_{1,n+1} > x_{1,n} > x_{2,n+1} > x_{2,n} > \cdots > x_{n,n+1} > x_{n,n} > x_{n+1,n+1}. \tag{2.6}$$

Hence

$$x'_{0,n+1} > x_{1,n+1} > x_{1,n}, \text{ and } x'_{0,n} > x_{1,n} > x'_{1,n}.$$

ϕ'_n vanishes at $\pm\infty$, hence ϕ''_n vanishes at some points $\zeta_n > x'_{0,n}$ and $\gamma_n < x'_{n,n}$. Now, from (1.3) and (1.4)

$$\phi''_n + (2n+1-x^2)\ \phi_n = 0\ ; \tag{2.7}$$

it follows that ϕ''_n has $n+2$ roots

$$\zeta_n = \sqrt{2n+1} > x_{1,n} > x_{2,n} > \dots > x_{n,n} > -\sqrt{2n+1} = \gamma_n.$$

Moreover,

$$x'_{0,n} < \sqrt{2n+1}\,, -\sqrt{2n+1} < x'_{n,n}. \tag{2.8}$$

For a sharper estimate of the roots $x_{\nu,n}$ cf. [3, p. 128].

It is seen from (1.1) that for all large values of x, $H_n(x) > 0$, hence $\phi_n(x) > 0$ for $x > x_{1,n}$, and $\phi_n(x)$ is decreasing for all $x > x'_{0,n}$, so that $\phi'_n(x) < 0$ for $x > x'_{0,n}$. Furthermore $\phi_n(x) < 0$ in the interval $x_{1,n} > x > x_{2,n}$, and $\phi_n(x) > 0$ for $x_{2,n} > x > x_{3,n}$, and so on, so that

$$\operatorname{sgn} \phi_n(x) = (-1)^r \text{ for } x_{r,n} > x > x_{r+1,n},\ r = 0, 1, 2, \dots, n, \tag{2.9}$$

where $x_{0,n} = +\infty$, $x_{n+1,n} = -\infty$. Similarly $\phi'_n(x) > 0$ for $x'_{0,n} > x > x'_{1,n}$, and in general

$$\operatorname{sgn} \phi'_n(x) = (-1)^{r-1} \text{ for } x'_{r-1,n} > x > x'_{r,n},\ r = 0, 1, 2, \dots, n+1 \tag{2.10}$$

where $x'_{-1,n} = +\infty$, $x'_{n+1,n} = -\infty$.

3. From (1.15)

$$\phi^2_{n+1}(x'_{r,n+1}) = \frac{(\phi'_n(x'_{r,n+1}))^2}{2(n+1) - (x'_{r,n+1})^2} + \phi^2_n(x'_{r,n+1}),$$

or

$$\mu^2_{r,n+1} = g(x'_{r,n+1}), \tag{3.1}$$

where

$$g(x) = \phi^2_n(x) + \frac{(\phi'_n(x))^2}{2(n+1)-x^2}. \tag{3.2}$$

Thus

$$g'(x) = 2\phi_n\phi'_n + \frac{2\phi'_n\phi''_n}{2(n+1)-x^2} + \frac{2x(\phi'_n(x))^2}{[2(n+1)-x^2]^2}$$

$$= 2\phi'_n\left\{\phi_n + \frac{\phi''_n}{2(n+1)-x^2} + \frac{x\phi'_n}{[2(n+1)-x^2]^2}\right\},$$

so that, using (2.7)

$$g'(x) = 2\phi'_n\left\{\phi_n - \frac{2n+1-x^2}{2n+2-x^2}\phi_n + \frac{x\phi'_n}{[2(n+1)-x^2]^2}\right\}$$

$$= 2\phi'_n\left\{\frac{\phi_n}{2n+2-x^2} + \frac{x\phi'_n}{(2n+2-x^2)^2}\right\}$$

$$= \frac{2\phi'_n}{2n+2-x^2}\left\{\phi_n + \frac{x\phi'_n}{2n+2-x^2}\right\}$$

$$= \frac{2\phi'_n}{(2n+2-x^2)^2}[(2n+2-x^2)\phi_n + x\phi'_n].$$

Differentiating the second equation of (1.11) and using (2.7) we get

$$\phi''_n = (x^2-2n-1)\phi_n = \phi_n + x\phi'_n - \sqrt{2(n+1)}\,\phi'_{n+1};$$

hence,

$$x\phi'_n + (2n+2-x^2)\,\phi_n = \sqrt{2n+2}\,\phi'_{n+1}, \qquad (3.3)$$

so that

$$g'(x) = \frac{2\sqrt{2n+2}\,\phi'_n\phi'_{n+1}}{(2n+2-x^2)^2}. \qquad (3.4)$$

From (3.3) for $x = x'_{r,n} (0 \leqslant r \leqslant n)$

$$(n+2-x'^2_{r,n})\phi_n = \sqrt{2n+2}\,\phi'_{n+1}; \qquad (3.5)$$

by (2.8) $2n+2-x'^2_{r,n} > 0$, by (2.5) $x_{r,n} > x'_{r,n} > x_{r+1,n}$ so that by (2.9) $\operatorname{sgn}\phi_n(x'_{r,n}) = (-1)^r$, whence $\operatorname{sgn}\phi'_{n+1}(x'_{r,n}) = (-1)^r$. Similarly for $x = x'_{r+1,n}$ $(-1 \leqslant r \leqslant n-1)$ $\operatorname{sgn}\phi_n(x'_{r+1,n}) = (-1)^{r+1}$, so that $\operatorname{sgn}\phi'_{n+1}(x'_{r+1,n}) = (-1)^{r+1}$. It follows that ϕ'_{n+1} vanishes at some point in the interval $x'_{r,n} > x > x'_{r+1,n}$, $r = 0, 1, \ldots, n-1$. Furthermore for $x > \sqrt{2n+2}$ the left side of (3.5) is negative, hence $\phi'_{n+1}(x) < 0$, while $\phi'_{n+1}(x'_{0,n}) > 0$; thus the largest zero of ϕ'_{n+1} is $> x'_{0,n}$. We now have the inequalities

$$x'_{0,n+1} > x'_{0,n} > x'_{1,n+1} > x'_{1,n} > \ldots > x'_{n+1,n+1}.$$

4. We now consider for a fixed r, $g'(x)$ in the interval $x'_{r,n+1} > x > x'_{r,n}$. From (2.10) sgn $\phi'_n(x) = (-1)^r$ for $x'_{r-1,n} > x > x'_{r,n}$, and sgn $\phi'_{n+1}(x) = (-1)^{r-1}$ for $x'_{r,n+1} > x > x'_{r+1,n+1}$. Furthermore $x'_{r-1,n} > x'_{r,n+1} > x'_{r,n} > x'_{r+1,n+1} > x'_{r+1,n}$; it follows from (3.4) that $g'(x) < 0$ for $x'_{r,n} < x < x'_{r,n+1}$, so that $g(x)$ is decreasing in that interval, hence

$$g(x'_{r,n}) > g(x'_{r,n+1}),$$

thus, using (3.2) and (3.1)

$$\mu^2_{r,n} > \mu^2_{r,n+1}, \ n \geqslant r \geqslant 0.$$

This proves (1.6). It follows from (1.5) and (1.6) that in the triangular matrix $(\mu_{r,n})$, $0 \leqslant r \leqslant n$, the largest term in each column and in each row is the first term, hence

$$|\phi_n(x)| \leqslant \max |\phi_0(x)| = \phi_0(0) = 1.$$

This proves a conjecture of G. E. H. Reuter [on the boundedness of the Hermite orthogonal system, *Journal of the London Math. Society*, 24 (1949), 159-163], who gave the bound $2^{1/4}$, while uniform boundedness was first proved by E. Hille [*Annals of Math.* 27 (1926), 435].

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EINE BEMERKUNG ÜBER PRODUKTE AUFEINANDER FOLGENDER ZAHLEN

Von Richard Oblath

In Unkenntnis der Arbeit, On some Diophantine equations, von S. S. Pillai, *Journal of the Indian Mathematical Society*, 18 (1930), 291-295, *Jahrb. ü. d. Fortschr. d. Math.* 56/II (1930), 878-879, war ich auf Grund des Referates im *Jahrbuch ü.d. Fortschritte der Mathematik* der Meinung, dass er die Unmöglichkeit der diophantischen Gleichung

$$x^{nr} \pm 1 = ry^{n}$$

auf Grund des grossen Fermat-schen Satzes bewiesen hat; ich publizierte daher einen hypothesenfreien Beweis dieses Satzes in der polnischen Zeitschrift *Wiadonoczny Matematiczne* 42 (1936), 127-128. Als ich den Separatabdruck der Pillai-schen Arbeit erhielt, ersah ich, dass sein Beweis sich auf nichts Unbewiesenes stützt, sondern vollkommen korrekt ist; mein Beweis ist im Wesen derselbe als der Pillai-sche. Meine Arbeit war daher überflüssig. Diesen Tatbestand habe ich in einem in 1940 erschienen Aufsatze dargestellt (Cf. *Mat. es. Fiz. Lapok*, 47 (1940), 57-76, spez. p. 66). Da die betreffende Arbeit nur in ungarischer Sprache erschienen ist, betrachte ich es als eine Ehrenpflicht, dies auch in Indien zu bekennen.

Die Frage, ob das Produkt beliebig vieler aufeinander folgender Zahlen eine volle Potenz darzustellen vermag, oder nicht, ist schon zweihundert Jahre alt. [Die Vermutung scheint—wenigstens in den einfachsten Fallen—vom berühmten russischen Mathematiker Goldbach zu stammen. [2], p. 679]. Von einigen—hauptsächlich von Liouville herrührenden—speziellen Sätzen abgesehen, hat die Lösung unseres Problemes bis 1933 fast gar keinen Fortschritt erzielt. Seit 1933 hat man einige sehr bemerkenswerte und interessante Resultate erhalten, das Problem ist aber noch immer sehr weit von seiner vollständigen Lösung entfernt. Die Zusammenstellung der bekannten Ergebnisse und die ausführliche Bibliographie sind in meinen Arbeiten [5]

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und [6] enthalten. [Die Zahlen in eckigen Klammern beziehen sich auf die Bibliographie am Ende dieser Arbeit.]

Herr Georg Szekeres hat eine ganz neue Idee in die Behandlung unseres Problemes eingeführt. [Briefliche Mitteilung an den Verfasser.] Seine Methode gründet sich auf die Bemerkung, dass das Produkt aufeinander folgender Zahlen keine volle Potenz werden kann, wenn eine dieser Zahlen zu allen Übrigen teilerfremd ist. Herr Szekeres hat seinen Beweis nicht publiziert, es ist daher wünschenswert seinen schönen Beweis hier kurz zu skizzieren.

Wir wollen also voraussetzen, dass das Produkt der consekutiven Zahlen

$$x,\ x+1,\ \dots,\ x+k-1 \qquad (1)$$

eine volle n-te Potenz ist, und, dass eine derselben z. B. x_0. zu allen Übrigen teilerfremd its. Dann ist x_0 unbedingt eine volle n-te Potenz $x_0 = a^n$ und es ist

$$(a^n-1)\ (a^n-2)\dots(a^n-k_1)\ (a^n+1)\dots(a^n+k_2) = y^n$$
$$(k_1+k_2 = k-1)$$

$y = a^{k-1}$ ist auszuschliessen, weil doch y durch a^n-1 teilbar ist und $(a, a^n-1) = 1$. Wenn wir also beweisen, dass

$$(a^{k-1}-1)^n < (a^n-1)\dots(a^n-k_1)\ (a^n+1)\dots(a^n+k_2) < (a^{k-1}+1)^n \qquad (2)$$

d. h.

$$a^{k-1}-1 < y < a^{k-1}+1$$

gilt, dann ist unsere Behauptung vollkommen bewiesen.

Offenbar ist

$$(a^n-1)\dots(a^n-k_1)\ (a^n+1)\dots(a^n+k_2) \leqslant (a^n+1)\dots(a^n+k-1).$$

Um also die zweite Hälfte von (2) zu beweisen, genügt der Nachweis der Ungleichheit

$$(a^n+1)\dots(a^n+k-1) < (a^{k-1}+1)^n. \qquad (3)$$

Zuerst sei $n = k$ und $a \geqslant k-1$, dann hat man

$$(a^n+1)\ldots(a^n+k-1) \leqslant (a^n+a)(a^n+a)\ldots(a^n+a)$$
$$= (a^n+a)^{n-1} = (a^{n-1}+1)^{n-1}a^{n-1} < (a^{n-1}+1)^n.$$

Um die Ungleichheit auch auf $n > k$ auszudehnen, bedienen wir uns der vollständigen Induktion. Wir nehmen also an, dass (3) für n-1 schon bewiesen ist, dass also

$$(a^{n-1}+1)\ldots(a^{n-1}+k-1) < (a^{k-1}+1)^{n-1}$$

wirklich gilt; daraus ergibt sich

$$(a^n+1)\ldots(a^n+k-1) < (a^n+a)\ldots(a^n+a(k-1))$$
$$< (a^{k-1}+1)^{n-1}a^{k-1} < (a^{k-1}+1)^{n-1}(a^{k-1}+1)$$
$$= (a^{k-1}+1)^n;$$

unsere Behauptung ist also bewiesen. Ein ähnlicher Gedankengang beweist auch die erste Hälfte der Ungleichheit (2). Der Fall $a < k-1$ ist komplizierter; der Beweis beruht aber auf derselben Grundlage und verläuft ähnlich. Wir unterlassen die detaillierte Diskussion, da wir uns in der vorliegenden Arbeit nur mit $k \leqslant 17$ befassen, ein Fall, den wir in der Arbeit [6] erledigt haben.

Ursprünglich dachte man, dass durch die soeben dargestellte Methode von Szekeres das Problem vollständig erledigt sei. Bald stellte sich aber heraus, dass die Voraussetzung, wonach es unter k consekutiven Zahlen immer eine zu allen übrigen teilerfremdc gibt, nur für $k \leqslant 16$ allgemein gilt, denn für $k = 17$ gibt es schon zwei Ausnahmen, nämlich

$$x \equiv 2184 \mod M = 2.3.5.7.11.13$$

und $\quad x \equiv 27830 \mod M.$

Im Jahre 1940 hat dann S. Sivasankaranarayana Pillai [[9], [10], [11]. Ich kenne diese Arbeiten nur aus den Referaten in der Math. Reviews und im Zentralblatt f. Math.] bewiesen, dass es für jedes $17 \leqslant k \leqslant 430$ solche k aufeinanderfolgende Zahlen gibt, unter denen keine einzige zu allen übrigen teilerfremd ist. Herr A.

Brauer hat dann dieses Resultat weiter verschärft, [1] indem er zeigte, dass es für *jedes* $k \geqslant 17$ gilt, so, dass die Szekeres-sche Methode nicht weiter fortgesetzt werden kann.

Nun sind im Falle $k = 17$ eben diese beiden Ausnahmefalle, in denen die Methode von Szekeres versagt, so gestaltet, dass die Vereinigung der beiden in meiner Arbeit[5] benützten Methoden zum Ziele führt. Es gilt also der

Satz. *Das Produkt von höchstens 17 aufeinander folgenden ganzen rationalen nicht verschwindenden Zahlen ist keine volle Potenz, d. h. die diophantische Gleichung*

$$x(x+1)\dots(x+k-1) = y^n \tag{I}$$

ist für $k \leqslant 17$, $n > 1$ *und* $y \neq 0$ *unmöglich.*

Wenn $k \leqslant 16$ ist, ergibt den Beweis die Methode von Szekeres. Wir haben uns also nur mit den beiden schon erwähnten Ausnahmewerten $x \equiv 2184$ und $x \equiv 27830$ mod M zu befassen wobei [9]

$$M = 2 \prod_{3 \leqslant p < k} p = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13.$$

Es sind mod $2M$ vier Fälle zu unterscheiden, und zwar $x_1 \equiv 2184$, $x_2 \equiv 2184 + M$, $x_3 \equiv 27830$, $x_4 \equiv 27830 + M$ mod $2M$. In jedem dieser Fälle kann man in der Folge (1) zwei solche Zahlen finden, welche mit den übrigen Zahlen der Folge x, $x+1, \dots, x+k-1$ keinen gemeinsamen ungeraden Primteiler p (d. h. $3 \leqslant p < k$) besitzen und deren grösster gemeinsamer Teiler die Zahl 2 ist. Z. Beispiel 2186 und 2194; 27838 und 27842; $2188+M$ und $2192+M$, endlich $27836+M$ und $27844+M$. Wenn (I) besteht, dann haben diese beiden Zahlen immer die Form $2u^n$. Im ersten Falle hat man $x_1 + 2 = 2u_1^n$: $x_1 + 10 = 2u_2^n$ d. h. $u_2^n - u_1^n = 4$ was doch offenbar unmöglich ist. In den übrigen Fallen erhalten wir ebenfalls Gleichungen, deren Unmöglichkeit in die Augen springt, und zwar $u_2^n - u_1^n = 2$, $u_2^n - u_1^n = 2$, beziehungsweise $u_2^n - u_1^n = 4$, Q. e. d.

Unsere Methode (wobei der gemeinsame Teiler natürlich auch eine andere Zahl sein kann) ist auch in anderen Fällen anwendbar, zur allgemeinen Lösung des Problemes ist sie aber nicht kräftig genug.

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ON A SERIES OF EIGENFUNCTIONS

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1. Introduction. Let us consider the boundary value problem

$$\triangle u+\lambda u \equiv \sum_{i=1}^{k} \frac{\partial^2 \mu}{\partial x_i^2} + \lambda u = 0$$

$$u(0, x_2, x_3, \ldots, x_k) = u(2\pi, x_2, \ldots, x_k)$$

$$u(x_1, 0, x_3, \ldots, x_k) = u(x_1, 2\pi, x_3, \ldots, x_k)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$u(x_1, \ldots, x_{k-1}, 0) = u(x_1, \ldots, x_{k-1}, 2\pi) \qquad (1)$$

in the domain $D \equiv (0 \leqslant x_1, x_2, \ldots, x_k \leqslant 2\pi)$ in the k-dimensional Euclidean space. The eigenvalues are the numbers λ_n which are such that $\lambda_n = n_1^2 + n_2^2 + \cdots + n_k^2$, where $n_1, n_2, \ldots, n_k$ are positive or negative integers or zero, and the associated eigenfunctions are ω_n, where $\omega_n = e^{i(n_1 x_1 + n_2 x_2 + \cdots + n_k x_k)}$. This fact, that numbers which can be expressed as sums of squares of integers happen to be eigenvalues in a particular eigenvalue problem, raises the question whether the known results in the theory of representation of numbers as sums of squares of integers, also known as lattice-point theory, can be extended to eigenvalues and eigenfunctions in general.

In fact, Dr. S. Minakshisundaram, in the course of a recent lecture* considered this question, developed an analogy between the two types of problems and discussed its scope.

Bochner and Chandrasekharan [1] considered the summability of the series

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$$\sum_{n=1}^{\infty} r_k(\lambda_n\, h)\, J_{\beta+k/2}(2\pi\xi\sqrt{\lambda_n})\, \lambda_n^{-s},$$

where $r_k(\lambda_n, h) = \sum_{n_1^2+n_2^2+\ldots n_k^2=\lambda n} e^{2\pi i(n_1 h_1+n_2 h_2+\ldots n_k h_k)}$

and $J_{\beta+k/2}$ is the Bessel function of order $\beta+k/2$. In particular, in the special case $\beta=-1$, they have shown that [1, 4.4]

"If $\xi^2 \neq (n_1+h_1)^2+(n_2+h_2)^2+\ldots+(n_k+h_k)^2$,

then

$$\sum_{n=1}^{\infty} r_k(\lambda_n, h)\, J_{k/2-1}(2\pi\xi\sqrt{\lambda_n})\, \lambda_n^{-s-k/4+1/2}$$

is summable $(R;\lambda,\gamma)$ for $\gamma > -2s+\frac{1}{2}(k-1)$ and the sum is zero for $s=-1, -2, -3 \ldots$." The above series represents an entire function of s with zeros at $s=-1, -2, -3, \cdots$

We shall consider the following boundary-value problem in a bounded domain D, with a regular boundary B, in the k-dimensional Euclidean space $(k>1)$.

$$\triangle u+\lambda u \equiv \sum_{i=1}^{k} \frac{\partial^2 u}{\partial x_i^2}+\lambda u=0,$$

$$\text{I} \qquad u(x) \equiv u(x_1, x_2, \ldots, x_k)=0 \text{ on } B, \tag{2}$$

$$\text{or II} \qquad \frac{\partial}{\partial n} u(x)=0 \text{ on } B,$$

where $\partial/\partial n$ denotes the derivative along the normal to the boundary B. This boundary value problem has an infinite number of non-negative eigenvalues and eigenfunctions. We assume that the eigenvalues are arranged in non-decreasing order and that the eigenfunctions constitute a complete and ortho-normal set. The mth eigenvalue and the mth eigenfunction are respectively denoted by λ_m and ω_m. In the case of the boundary condition II, $\lambda_0=0$ is a simple eigenvalue with the eigenfunction $\omega_0=V^{-1/2}$, where V is the volume of the domain D, while in the case of

the boundary condition I, all the eigenvalues are positive. If we suppose that the suffix m runs through 1, 2, 3, ... in the case I, and through 0, 1, 2, ... in the case II, we have in both cases $\lambda_1 > 0$.

We define a function $f(s)$ of the complex variable $s = (\sigma + i\tau)$ in the following manner.*

$$f(s) = \sum_{m=1}^{\infty} \frac{\omega_m(P)\,\omega_m(Q)}{\lambda_m^{\,s+k/4-1/2}}\, J_{k/2-1}\,(R\sqrt{\lambda_m}), \tag{3}$$

where P and Q are two points in the domain D. The series in (3) is the analogue of the one considered by Bochner and Chandrasekharan [1, 4.4]. This series is absolutely convergent only for large values of σ, say $\sigma > \sigma_0$. Hence $f(s)$ as represented by the series is an analytic function in the region $R(s) > \sigma_0$. The aim of this note is to consider the analytic extension of $f(s)$ all over the complex plane. We shall be concerned only with the case $k > 1$, which is of interest to us, since in the case $k = 1$, the series (3) reduces to one involving trigonometric functions only and can be treated using known properties of the Riemann Zeta functions. The following theorems are proved in this note.

THEOREM I. *P and Q are distinct points in D, d is the distance between them and R is such that the sphere with centre P and radius R is completely contained in D. If $R \neq d$, the analytic continuation of $f(s)$ is an entire function of s with zeros at $s = 0, -1, -2, \ldots$, in the case I of the boundary condition and at $s = -1, -2, -3, \ldots$, in the case II of the boundary condition.*

If $R = d$, we have to distinguish between the cases k odd and k even.

THEOREM II. *If $R = d$, R is as in Theorem I and k is odd, the analytic extension of $f(s)$ is a meromorphic function of s with a finite number of simple poles at $s = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \ldots, (\frac{1}{2}-m), \ldots, (\frac{1}{2}-\frac{1}{2}(k-3))$; the residue at $s = \frac{1}{2}-m$, being*

$$\Big\{(-1)^m \left(\frac{k-3}{2}+m\right)!/2^{(k+2)/2}\pi^{\frac{k+1}{2}}\Gamma(\tfrac{1}{2}-m)\, m! \times \left(\frac{k-3}{2}-m\right)!\, R^{2m+k/2}\Big\}.$$

This function has its zeros at $s=0, -1, -2 \ldots$, *in case I and at* $s=-1, -2, \ldots$ *in case II of the boundary condition.*

If, however, k is even, the function is a meromorphic function of s with an infinite number of simple poles at $s=\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \ldots$, the residue at $s=\frac{1}{2}-m$, being the same as above and with zeros at the same points as above.

THEOREM III. *If P and Q coincide and R is as in Theorem I, the analytic continuation of the function represented by the series*

$$\sum_{m=1}^{\infty} \frac{\omega_m{}^2(P)}{\lambda_m{}^{s+k/4-1/2}} J_{k/2-1}\,(R\sqrt{\lambda_m})$$

represents an entire function of s with zeros at $s=0, -1, -2, \ldots$ *in case I and at* $s=-1, -2, \ldots$ *in case II of the boundary condition.*

We observe that the case $R=d$, which corresponds to $\xi^2=\sum_{i=1}^{k}(n_i+h_i)^2$, excluded in [1] is considered in Theorem II.

2. **Preliminaries.** We require the following results.

$$\sum_{m=1}^{\infty} \omega_m(x)\,\omega_m(y)\; e^{-\lambda_m t} = G(x, y;\, t) - \delta/V, \tag{4}$$

where $G(x, y, t)$ is the Green's function of the heat equation

$$\Delta u - \frac{\partial u}{\partial t} = 0$$

associated with either of the boundary condtions and satisfying the initial condition $\lim_{t=0} G(x, y, t) = 0$, x, y in D, $x \neq y$ and $\delta = 0$ or 1 according as the boundary condition I or II is assumed.

$$G(x, y\ t) = \frac{1}{2^k(\pi t)^{k/2}} \exp\left\{-\sum_{i=1}^{k} \frac{(x_i - y_i)^2}{4t}\right\} - g(x, y;\ t) \quad (5)$$

$$|g(x, y, t)| < \frac{c}{t^{k/2}} \exp(-l_y^2/t) \quad (6)$$

in any finite interval of t, for all x in D, where l_y is the minimum distance between y and points on B. c is a constant independent of x and t. [3, lemma]

$$\frac{1}{\Omega}\iint_{\Omega} \omega\, d\Omega = \frac{\omega_0 \Gamma(k/2)\, J_{k/2-1}(R\sqrt{\lambda})}{\left(\frac{R\sqrt{\lambda}}{2}\right)^{k/2-1}}. \quad (7)$$

where Ω is the surface area of the k-dimensional sphere $x_1^2+x_2^2 \ldots\ldots + x_k^2 = R^2$; ω_o is the value, at the origin, of the eigenfunction ω, associated with the eigenvalue λ. [2, 261]

$$I_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})} \int_0^{\pi} e^{-z\cos\theta} \sin^{2\nu}\theta\, d\theta, \quad (8)$$

where $I_\nu(z)$ is the Bessel function with imaginary argument and is of the order ν. [4, 79]

$$I_{n+1/2}(z) = \frac{1}{\sqrt{2\pi z}}\left[e^z \sum_{r=0}^{n} \frac{(-1)^r (n+r)!}{r!\,(n-r)!\,(2z)^r}\right.$$
$$\left. + (-1)^{n+1} e^{-z} \sum_{r=0}^{n} \frac{(n+r)!}{r!\,(n-r)!\,(2z)^r}\right] \quad (9)$$

where n is a positive integer. [4, 80]

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{m=0}^{\infty} \frac{(-1)^m (\nu, m)}{(2z)^m} + \frac{e^{-z+(\nu+1/2)\pi i}}{\sqrt{2\pi z}} \sum_{m=0}^{\infty} \frac{(\nu, m)}{(2z)^m}, \quad (10)$$

where $(\nu, m) = \dfrac{\{4\nu^2 - 1^2\}\{4\nu^2 - 3^2\} \ldots\ldots \{4\nu^2 - (2m-1)^2\}}{2^{2m}\, m!}$

if $m \neq 0$, and $= 1$, if $m = 0$. [4, 203]

3. **Proof of Theorem I.** In (4) let us take for y a fixed point Q in the domain D, and for x any point on the surface of the k-dimensional sphere, completely contained in D, with its centre at P and radius R and

take the average of both sides of (4) over the entire surface of the sphere. We get from the left side of (4), in view of (7),

$$\sum_{m=1}^{\infty} \frac{\omega_m(P)\,\omega_m(Q)\,\Gamma(k/2)\,e^{-\lambda_m t}\,J_{k/2-1}(R\sqrt{\lambda_m})}{\left(\frac{R\sqrt{\lambda_m}}{2}\right)^{k/2-1}} \tag{11}$$

since the series is uniformly convergent for $t > 0$. From (4) and (5) introducing polar coordinates, we obtain from the right side of (4)

$$\frac{1}{\Omega(2\sqrt{\pi t})^k}\, e^{\frac{-(R^2+d^2)}{4t}}$$
$$\times \iint_\Omega \cdots \int e^{\frac{Rd\cos\theta_1}{2t}} \sin^{k-2}\theta_1 \sin^{k-3}\theta_2 \ldots \sin\theta_{k-2} R^{k-1}\, d\theta_1 \ldots d\theta_{k-1}$$
$$+\frac{1}{\Omega}\iint_\Omega g(z, Q;\, t)\, d\Omega - \frac{\delta}{V} \tag{12}$$

where d is the distance between P and Q and z is any point on the sphere. Using (8), from (II) and (12) after some simplification we have

$$\sum_{m=1}^{\infty} \frac{\omega_m(P)\,\omega_m(Q)\,e^{-\lambda_m t}}{(R\sqrt{\lambda_m})^{k/2-1}} J_{k/2-1}(R\sqrt{\lambda_m})$$
$$= \frac{1}{(2\sqrt{\pi t})^k}\, e^{\frac{-(R^2+d^2)}{4t}} \cdot \frac{I_{k/2-1}\left(\frac{Rd}{2t}\right)}{\left(\frac{Rd}{2t}\right)^{k/2-1}} \tag{13}$$
$$+\frac{2^{1-k/2}}{\Omega\,\Gamma(k/2)}\left[\iint_\Omega g(z, Q;\, t)\, d\Omega - \frac{\delta\,\Omega}{V}\right].$$

If we now multiply both sides of (13) by t^{s-1} and integrate with respect to t from zero to infinity, from the left side of (13) we get

$$\Gamma(s) \sum_{m=1}^{\infty} \frac{\omega_m(P)\,\omega_m(Q)}{R^{k/2-1}\,\lambda_m^{s+k/4-1/2}} J_{k/2-1}(R\sqrt{\lambda_m}) \tag{14}$$

provided σ is sufficiently large. Denoting the right side

of (13) by L, to evaluate $\int_0^{\infty} Lt^{s-1}\,dt$, we split up the interval into two parts 0 to γ ($\gamma > 0$), and γ to ∞, and write

$$\int_0^{\infty} Lt^{s-1}\,dt = \int_0^{\gamma} Lt^{s-1}\,dt$$

$$+\int_{\gamma}^{\infty} \sum_{m=1}^{\infty} \frac{\omega_m(P)\,\omega_m(Q)\,e^{-\lambda_m t}}{(R\sqrt{\lambda_m})^{k/2-1}} J_{k/2-1}(R\sqrt{\lambda_m})\,t^{s-1}\,dt. \qquad (15)$$

The second integral on the right of (15) can be written as

$$\sum_{m=1}^{\infty} \frac{\omega_m(P)\,\omega_m(Q)}{(R\sqrt{\lambda_m})^{k/2-1}} J_{k/2-1}(R\sqrt{\lambda_m}) \int_{\gamma}^{\infty} e^{-\lambda_m t}\,t^{s-1}\,dt. \qquad (16)$$

This series is an entire function of s. The first integral on the right of (15) is equal to

$$\frac{1}{(2\sqrt{\pi})^k} \int_0^{\gamma} \frac{e^{-(R^2+d^2)/4t}\,t^{s-k/2-1}}{\left(\frac{Rd}{2t}\right)^{k/2-1}} I_{k/2-1}\left(\frac{Rd}{2t}\right) dt \qquad (17)$$

$$+\frac{2^{1-k/2}}{\Omega\,\Gamma(k/2)} \int_0^{\gamma}\int_{\Omega} g(z, Q;\, t)\,t^{s-1}\,dt\,d\Omega - \frac{2^{1-k/2}}{\Gamma(k/2)} \int_0^{\gamma} \frac{\delta t^{s-1}}{V}\,dt.$$

The second integral of (17), in view of (6) is majorized by

$$\frac{2^{1-k/2}}{\Gamma(k/2)} \int_0^{\gamma} c\,e^{-l^2 Q/4t}\,t^{\sigma-k/2-1}\,dt \qquad (18)$$

which remains finite for all values of σ so long as $l_Q \neq 0$, i.e. Q is an interior point. Therefore the second integral of (17) is an entire function of s. The third integral of (17) is equal to

$$\frac{-2^{1-k/2}\,\delta\,\gamma^s}{\Gamma(k/2)\,V s} \qquad (19)$$

provided $\sigma > 0$. For sufficiently small γ, using the asymptotic expansion (10), the first integral of (17) can be written as

$$\frac{1}{2^{(k+2)/2}\,\pi^{(k+1)/2}\,(Rd)^{(k-1)/2}} \int_0^{\gamma} e^{-\frac{(R-d)^2}{4t}}\,t^{s-3/2}\left\{1 + O(t)\right\} dt. \qquad (20)$$

Now if $R \neq d$, i.e. if the sphere described about P as centre does not contain Q on its surface, the integral in

(20) is an entire function of s. From (14) to (20), we have

$$\sum_{m=1}^{\infty} \frac{\omega_m(P)\,\omega_m(Q)}{\lambda_m^{\,s+k/4-1/2}} J_{k/2-1}\,(R\sqrt{\lambda_m})$$
$$= \frac{E(s)}{\Gamma(s)} - \frac{R^{k/2-1}\,\delta\,\gamma^s}{2^{k/2-1}\sqrt{\Gamma(k/2)}\,\Gamma(s+1)}, \qquad (21)$$

where $E(s)$ is an entire function of s. Hence $f(s)$ of (3), which is represented by the series on the left of (21) for large values of σ is now equal to the function on the right of (21). This function on the right of (21) is the analytic extension of $f(s)$ for all values of s and is an entire function of s with zeros at $s=0, -1, -2, \ldots$ in case I of boundary condition since δ is equal to 0 in this case. These zeros are the poles of $\Gamma(s)$. In case II of the boundary condition, since $\delta = 1$, the function does not vanish for $s=0$, but is equal to $-\dfrac{R^{k/2-1}}{2^{k/2-1}\sqrt{\Gamma(k/2)}}$ and hence the zeros are at $s=-1, -2, \ldots$. Thus the first theorem is proved.

4. **Proof of Theorem II.** If $R=d$, the integral in (20) is no longer an entire function of s. However, it can be analytically continued over the entire complex plane. As already mentioned in the introduction, we have to consider, in this case, k-odd and k-even separately. First we shall take up the case k-odd. Using (9), the first integral of (17) can be written as

$$\frac{1}{2^{(k+2)/2}\,\pi^{(k+1)/2}\,R^{k-1}}\Bigg[\int_0^{\gamma} \sum_{r=0}^{(k-3)/2} \frac{(-1)^r \left(\frac{k-3}{2}+r\right)!\; t^{s+r-3/2}}{r!\left(\frac{k-3}{2}-r\right)!\,R^{2r}}\,dt$$
$$+\int_0^{\gamma} (-1)^{(k-1)/2}\, e^{-R^2/t} \sum_{r=0}^{(k-3)/2} \frac{\left(\frac{k-3}{2}+r\right)!\; t^{s+r-3/2}}{r!\left(\frac{k-3}{2}-r\right)!\,R^{2r}}\,dt\Bigg] \qquad (22)$$

since we are considering the case $R > 0$, the second integral in (22) is an entire function of s. After integration the first integral in (22) turns out to be

$$\frac{1}{2^{(k+2)/2}\pi^{(k+1)/2}}\left[\sum_{r=0}^{(k-3)/2}\frac{(-1)^r\left(\frac{k-3}{2}+r\right)!\left\{t^{s+r-1/2}\right\}_0^\gamma}{r!\left(\frac{k-3}{2}-r\right)!\,R^{2r+k-1}\left(s+r-\frac{1}{2}\right)}\right]. \quad (23)$$

Provided $\sigma > \frac{1}{2}$, (23) will be equal to

$$\frac{1}{2^{(k+2)/2}\pi^{(k+1)/2}}\left[\sum_{r=0}^{(k-3)/2}\frac{(-1)^r\left(\frac{k-3}{2}+r\right)!\,\gamma^{s+r-1/2}}{r!\left(\frac{k-3}{2}-r\right)!\,R^{2r+k-1}\left(s+r-\frac{1}{2}\right)}\right]. \quad (24)$$

The function in (24) is defined for all values of s and is the analytic extension of (23) for $\sigma \leqslant \frac{1}{2}$. This function is a meromorphic function of s with a finite number of simple poles at $s=\frac{1}{2}-r$, r going through the values $0, 1, \ldots, (k-3)/2$, the residue at $s=(\frac{1}{2}-r)$ being

$$\frac{1}{2^{(k+2)/2}\pi^{(k+1)/2}}\left[\frac{(-1)^r\left(\frac{k-3}{2}+r\right)!}{r!\left(\frac{k-3}{2}-r\right)!\,R^{2r+k-1}}\right].$$

From (14) to (19) together with (22), (23) and (24) we have

$$\sum_{m=1}^{\infty}\frac{\omega_m(P)\,\omega_m(Q)}{\lambda_m^{\,s+k/4-1/2}}\,J_{k/2-1}\left(R\sqrt{\lambda_m}\right)$$

$$=\frac{E(s)}{\Gamma(s)}+\frac{M(s)}{\Gamma(s)}-\frac{R^{k/2-1}\,\delta\,\gamma^s}{2^{k/2-1}\,v\,\Gamma(k/2)\,\Gamma(s+1)}, \quad (25)$$

where $E(s)$ is an entire function of s and $M(s)$ is a meromorphic function of s with a finite number of poles at $s=\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \ldots (\frac{1}{2}-(k-3)/2)$; the residue at $s=\frac{1}{2}-r$ being

$$\frac{1}{2^{(k+2)/2}\pi^{(k+1)/2}}\left[\frac{(-1)^r\left(\frac{k-3}{2}+r\right)!}{r!\left(\frac{k-3}{2}-r\right)!\,R^{2r+k/2}\,\Gamma\left(\frac{1}{2}-r\right)}\right].$$

The function on the right of (25) constitutes the analytic extension of $f(s)$. The statement with regard to zeros contained in Theorem II is easily seen to be true.

If k is even we have to use the formula (10), and the asymptotic expansion does not terminate but contains an infinite number of terms. Corresponding to (22) we have in this case

$$\frac{1}{2^{(k+2)/2}\pi^{(k+1)/2}R^{k-1}}\left[\int_0^\gamma t^{s-3/2}\left\{\sum_{r=0}^{p-1}\frac{(-1)^r(k/2-1,\ r)t^r}{R^{2r}}+O(t^p)\right\}dt\right.$$

$$\left.+\int_0^\gamma e^{-R^2/t+\pi i(k-1)/2}\,t^{s-3/2}\left\{\sum_{r=0}^{p-1}\frac{(k/2-1,r)t^r}{R^{2r}}+O(t^p)\right\}dt.\right] \quad (26)$$

The second integral in (26) is at once seen to be an entire function of s. After integration the first integral in (26) turns out to be

$$\frac{1}{2^{(k+2)/2}\pi^{(k+1)/2}R^{k-1}}\left[\sum_{r=0}^{p-1}\frac{(-1)^r(k/2-1,r)\left\{t^{s+r-1/2}\right\}_0^\gamma}{R^{2r}\ (s+r-\frac{1}{2})}\right.$$

$$\left.+\frac{O\left\{t^{\sigma+p-1/2}\right\}_0^\gamma}{(\sigma+p-\frac{1}{2})}\right]. \quad (27)$$

If $\sigma>\frac{1}{2}$, (27) is equal to

$$\frac{1}{2^{(k+2)/2}\pi^{(k+1)/2}R^{k-1}}\left[\sum_{r=0}^{p-1}\frac{(-1)^r\ (k/2-1,\ r)\ \gamma^{s+r-1/2}}{R^{2r}\ (s+r-\frac{1}{2})}\right.$$

$$\left.+\frac{O\left\{\gamma^{\sigma+p-1/2}\right\}}{\sigma+p-\frac{1}{2}}\right]. \quad (28)$$

This is a function of s defined over the region of the complex plane to the right of $\sigma=\frac{1}{2}-p$, with simple poles at $s=\frac{1}{2},-\frac{1}{2},-\frac{3}{2},\ldots,(3/2-p)$; and this is the analytic continuation of (27) in the strip of the plane $\frac{1}{2}-p<\sigma\leqslant\frac{1}{2}$. By taking as many terms as we please we can extend the function in (27) throughout the complex plane. Hence the analytic extension of (27) all over the plane is a meromorphic function of s with an infinite number of simple poles at $s=\frac{1}{2},-\frac{1}{2},-\frac{3}{2},\ldots.$ The residue at $s=\frac{1}{2}-r$

is easily identified with the corresponding expression in the case k odd. Hence the analytic extension of $f(s)$ in this case is a meromorphic function with an infinite number poles, the residue at the pole $s=\frac{1}{2}-r$ being given by the corresponding expression in the case k odd. The zeros coincide with those of the corresponding function in the case k odd. This proves Theorem II.

5. **Proof of Theorem III.** In case P and Q coincide, $d=0$. Using (4), (5) and (7) if we take the average over the k-dimensional sphere with its centre at P and radius R, we obtain

$$\sum_{m=1}^{\infty} \frac{\omega_m^2(P)\, e^{-\lambda_m t}\, J_{k/2-1}(R\sqrt{\lambda_m})\, \Gamma(k/2)}{\left(\frac{R\sqrt{\lambda_m}}{2}\right)^{k/2-1}} = \frac{1}{(2\sqrt{\pi t})^k} e^{-R^2/4t} + \frac{1}{\Omega}\iint_\Omega g(z, P; t)\, d\Omega - \frac{\delta}{V}. \quad (29)$$

If we multiply both sides of (29) by t^{s-1} and integrate with respect to t from 0 to ∞ we get

$$\Gamma(s) \sum_{m=1}^{\infty} \frac{\omega_m^2(P)\, J_{k/2-1}(R\sqrt{\lambda_m})}{\lambda_m^{s+k/4-1/4}} = \frac{R^{k/2-1}}{2^{k/2-1}\,\Gamma(k/2)} \int_0^\infty L'\, t^{s-1} dt, \quad (30)$$

where L' denotes the expression on the right of (29). Splitting the interval 0 to ∞ into two parts 0 to γ $(\gamma > 0)$ and γ to ∞, we can write

$$\int_0^\infty L'\, t^{s-1}\, dt = \int_0^\gamma L'\, t^{s-1} dt + \int_\gamma^\infty \sum_{m=1}^{\infty} \frac{\Gamma(k/2)\, \omega_m^2(P)\, e^{-\lambda_m t}\, J_{k/2-1}(R\sqrt{\lambda_m})\, t^{s-1} dt}{\left(\frac{R\sqrt{\lambda_m}}{2}\right)^{k/2-1}}. \quad (31)$$

The second integral of (31) is easily seen to be an entire function of s. The first integral of (31) is equal to

$$\int_0^\gamma L'\, t^{s-1} dt = \frac{1}{(2\sqrt{\pi})^k} \int_0^\gamma e^{-R/2 4t}\, t^{s-k/2-1} dt + \frac{1}{\Omega} \int_0^\gamma \int_\Omega g(z, P; t)\, t^{s-1} dt\, d\Omega - \frac{\delta}{V} \int_0^\gamma t^{s-1}\, dt. \quad (32)$$

The second and third integrals on the right of (32) are similar to those of (17) and can be dealt with similarly. The first integral on the right of (32) is an entire function of s since $R > 0$. Hence we have

$$\sum_{m=1}^{\infty} \frac{\omega_m^2(P)\, J_{k/2-1}(R\sqrt{\lambda_m})}{\lambda_m^{\,s+k/4-1/2}} = \frac{E(s)}{\Gamma(s)} - \frac{R^{k/2-1}\, \delta\, \gamma^s}{2^{k/2-1}\, V\, \Gamma(k/2)\, \Gamma(s+1)}, \quad (33)$$

where $E(s)$ is an entire function of s. From (33) the results stated in Theorem III at once follow.

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ON THE METRIC CHARACTERIZATION OF METRIC LATTICES

By DAVID ELLIS

1. Introduction. For accounts of the fundamental concepts from lattice theory and the topology of metric spaces used below, one may consult the books of Garrett Brikhoff and L. M. Blumenthal [1, 2]

By a metric lattice we mean a lattice L with meet $a \wedge b$, join $a \vee b$, and inclusion (in the broad sense) $a \subset b$, on which is defined a valuation (non-negative, real-valued function) $|a|$ so that:

(1) If $a \subset b$ and $a \neq b$, then $|a| < |b|$.

(2) For each a and b, $|a \vee b| + |a \wedge b| = |a| + |b|$.

We define distance by $\delta(a, b) = |a \vee b| - |a \wedge b|$. Under this definition of distance the point-set of L forms a metric space [3, 4] and we understand this definition of distance when referring to metric lattices.

Glivenko, and later, Smiley and Transue, have shown [4, 5] that a metric space is congruent to a metric lattice with first element if and only if it is almost ordered (defined shortly) with respect to one of its elements. In this note we show that every compact metric lattice (compact in the sense of Frechét which coincides with bicompactness in metric space) has a first (and last) element and that, hence, the almost ordering property is the complete characterization of metric lattices among compacta (compact metric spaces). We also deduce from a well-known result that every compact metric lattice is (lattice) complete.

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English letters refer to a metric lattice L, antique letters to an arbitrary metric space, and Greek letters to real numbers. We employ only the metric topology in L and the axiom of choice is implicit in the main theorem.

2. **Almost ordered metric spaces.** Let **M** be a metric space. One says that **b** is between **a** and **c** in **M** and writes **abc** if $\delta(\mathbf{a}, \mathbf{b})+\delta(\mathbf{b}, \mathbf{c}) = \delta(\mathbf{a}, \mathbf{c})$. (This is between-ness in the broad sense. Between-ness in the strict sense requires distinctness of the points and is more common in general metric geometry). **M** is said to be *almost ordered* with respect to one of its elements **f** if

(3) For each **a** and **b** in **M** there is an element **c** so that **acb**, **fca** and **fcb** subsist and **fxa** and **fxb** imply **fxc**.

(4) For each **a** and **b** in **M** there is an element **d** so that **adb**, **fad** and **fbd** subsist and **fay** and **fby** imply **fdy**.

The result obtained by Glivenko and by Smiley and Transue [4, 5] may be stated :

Lemma. *A metric space* **M** *is congruent to a metric lattice with first element if and only if* **M** *is almost ordered with respect to one of its elements.*

3. **Compact metric lattices.**

Theorem. *Every compact metric lattice possesses a first (and last) element.*

Proof. We shall show the existence of a first element. The existence of a last element follows dually. Let L be compact. By the Policeman's Theorem (Existence of ϵ-nets), for each $\epsilon > 0$ there is a finite subset $p_1^\epsilon, p_2^\epsilon, \ldots, p_{\eta}^{\epsilon}(\epsilon)$ so that for any x in L there is an α, $1 \leqslant \alpha \leqslant \eta(\epsilon)$, for which $\delta(x, p_\alpha^\epsilon) < \epsilon$.

Select such a finite set $p_1^{\epsilon_\alpha}, p_2^{\epsilon_\alpha}, \ldots, p_{\eta}^{\epsilon_\alpha}(\epsilon_\alpha)$ for each of a sequence of positive ϵ_α converging to 0. Denote $\bigwedge_{k=1}^{\alpha} \bigwedge_{\lambda=1}^{\eta(\epsilon_k)} (p_\lambda^{\epsilon_k})$ by p_α. Now, there is a convergent subsequence

(since L is compact) of the sequence $\{p_\alpha\}$ converging to a point of L. Denote any such point by f. It is clear that for each α, $f \subset p_\alpha$. But every metric lattice is a (sequentially) topological lattice in its metric topology and every point of L is the limit of a sequence of points of the double sequence $\{p_\alpha^{\varepsilon\beta}\}$. Hence $f \subset x$ for each x in L and f is the desired first element.

Combining this Theorem with the previously stated Lemma we obtain :

THEOREM. *A compactum is congruent to a metric lattice if and only if it is almost ordered with respect to one of its elements.*

PROOF. This follows immediately from the Theorem and Lemma since compactness is a metric invariant.

Now, in Garrett Birkhoff's book [1, Th. 15, Chap. IV], one finds that any metrically complete metric lattice is conditionally (lattice) complete. That is, lattice bounded subsets are (lattice) complete. Every compact metric lattice (even every compact metric space) is metrically complete and the first Theorem above shows that ever compact metric lattice is (lattice) bounded. Hence we infer :

THEOREM. *Every compact metric lattice is lattice complete.*

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ON UNBOUNDED OPERATORS IN HILBERT SPACE

By M. H. STONE

1. Introduction. A virtually indispensable device for the study of a non-bounded linear operator A in a Hilbert space $\mathfrak{H}$ consists in passing from the given operator to an associated family of bounded linear operators. As familiar examples of this device we may cite the characterization of a symmetric operator by its Cayley transform, and that of a self-adjoint operator by its resolvent, its Cayley transform, or its spectral family. In this paper* we shall discuss the correspondence between a quite *arbitrary* operator A and the two-rowed operator-matrix which defines the projection of the direct sum $\mathfrak{H} \oplus \mathfrak{H}$ onto the closed subspace generated by the graph of A. Since a closed linear operator is uniquely determined by this associated matrix, we may conveniently call it, even in the general case, the *characteristic matrix* of A. The use of the characteristic matrix is implicit in a paper of Tseng [1], where certain of its components provide a tool for the spectral analysis of a self-adjoint operator. Here it will enable us to give a more elementary derivation of the principal properties of non-bounded linear operators, first established by von Neumann ([1], [2], [3]). It will also enable us to prove, without any appeal to the spectral theorem, an interesting commutation criterion for an operator to have a normal extension. As this result, in a form only slightly less general, was first established by

* Substantially all the results presented here were obtained in the autumn of 1936, when the author was in residence at the Institute for Advanced Study, Princeton, as a Fellow of the John Simon Guggenheim Memorial Foundation. While much of it has been presented in lectures from time to time, a publishable version was not prepared until 1951.

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Sasaki and Ogasawara [1] in a joint paper and by Mimura [1] in an independent one, we shall refer to it as the S-O-M Theorem.

Here we shall permit the scalar system for $\mathfrak{H}$ to consist of the real numbers, the complex numbers, or the quaternions* ; and while requiring $\mathfrak{H}$ to be complete we shall impose no condition of separability. The standard notations and terminology of Hilbert space theory will be used without explicit references to the literature.

2. **The characteristic matrix.** An *arbitrary* operator A is characterized by its *graph* $\mathfrak{G} = \mathfrak{G}(A)$, defined as the set of all vectors (x, y) in the direct sum $\mathfrak{H} \oplus \mathfrak{H}$ such that $y = Ax$. A part of $\mathfrak{H} \oplus \mathfrak{H}$ is the graph of an operator if and only if for each x it contains at most one vector (x, y). An operator is linear or closed if and only if its graph is respectively a linear set or a closed set. Similarly A is a restriction of B, or B an extension of A, if and only if $\mathfrak{G}(A)$ is a part of $\mathfrak{G}(B)$. The involution $(x, y) \rightarrow (y, x)$ carries $\mathfrak{G}(A)$ into the set $((y, x); y = Ax)$, which we shall call the *inverted graph* of A. For A to have an inverse it is necessary and sufficient that its inverted graph be the graph of an operator, which is, of course, none other than the inverse in question. Each bounded linear operator S in $\mathfrak{H} \oplus \mathfrak{H}$ is uniquely expressible, through the relation

$$S: (x_1, x_2) \rightarrow (S_{11} x_1 + S_{12} x_2, S_{21} x_1 + S_{22} x_2), \qquad (1)$$

in terms of a two-rowed matrix $< S_{ik} >$ of bounded linear operators in $\mathfrak{H}$. Our fundamental definition can now be introduced:

Definition 1. *The matrix* $< P_{ik} >$ *of the projection* P *of* $\mathfrak{H} \oplus \mathfrak{H}$ *onto the closed subspace generated by the graph* $\mathfrak{G} = \mathfrak{G}(A)$ *is called the characteristic matrix of the operator* A.

* In the mimeographed version of the author's lectures of 1949-50 at the Tata Institute of Fundamental Research, Bombay, it may be seen how little the quaternionic case really differs from the more familiar real and complex cases.

Upon writing down in terms of the matrix $< P_{ik} >$ the identities $P^* = P$ and $P^2 = P$ characteristic for a projection, we obtain the following result.

THEOREM 1. *The elements of the characteristic matrix satisfy the relations*

$$(2)\quad P_{ik}^* = P_{ki}; \qquad \sum_{\alpha=1}^{2} P_{i\alpha} P_{\alpha k} = P_{ik}. \quad (3)$$

Any matrix of bounded linear operators in $\mathfrak{H}$ *which satisfies* (2) *and* (3) *represents a projection in* $\mathfrak{H} \oplus \mathfrak{H}$.

Concerning an arbitrary projection in $\mathfrak{H} \oplus \mathfrak{H}$ we note the following essential fact.

THEOREM 2. *If P is the projection of* $\mathfrak{H} \oplus \mathfrak{H}$ *onto a closed subspace* $\mathfrak{M}$, *then* $\mathfrak{M}$ *is the graph of an operator A if and only if*

$$P_{12}\, x = (I - P_{22})\, x = 0 \text{ implies } x = 0. \quad (4)$$

The operator A is necessarily a uniquely determined closed linear operator, which can be described as the mapping

$$A:\ P_{11}\, y_1 + P_{12}\, y_2 \to P_{21}\, y_1 + P_{22}\, y_2,$$

where y_1 *and* y_2 *are arbitrary vectors in* $\mathfrak{H}$. *In particular the identities*

$$(5)\quad P_{21} = AP_{11}; \qquad P_{22} = AP_{12} \quad (6)$$

are verified.

PROOF. Since the closed set $\mathfrak{M}$ is linear, the criterion for it to be a graph—necessarily that of a uniquely determined closed linear operator A—reduces to the simple condition that $(0, x)$ imply $x = 0$. Now $(0,x) \in \mathfrak{M}$ if and only if $(0, x) = P(0, x) = (P_{12}x, P_{22}x)$. Consequently for $(0, x) \in \mathfrak{M}$ to imply $x = 0$, it is necessary and sufficient that (4) hold. Since the vector $P(y_1, y_2) = (P_{11}y_1 + P_{12}y_2, P_{21}y_1 + P_{22}y_2)$ varies over $\mathfrak{M}$ when y_1 and y_2 vary over $\mathfrak{H}$, we can evidently describe A as the mapping indicated above. In particular the identities (5) and (6) follow at once when we put $y_2 = 0$ and $y_1 = 0$ in turn. Of course it is implicit in these statements that whenever two

vectors of the form $P_{11}y_1+P_{12}y_2$ are equal the associated vectors $P_{21}y_1+P_{22}y_2$ are also equal. However, it is of some interest to derive this result directly from the properties of the matrix $<P_{ik}>$. Since we are dealing with linear operators, it suffices to show that $P_{11}y_1+P_{12}y_2=0$ implies $P_{21}y_1+P_{22}y_2=0$. Taking $P_{11}y_1+P_{12}y_2=0$ and putting $P_{21}y_1+P_{22}y_2=x$, we apply (2) and (3) in the computations

$$\begin{aligned} P_{12}x &= P_{12}(P_{21}y_1+P_{22}y_2)+P_{11}(P_{11}y_1+P_{12}y_2) \\ &= (P_{11}P_{11}+P_{12}P_{21})y_1+(P_{11}P_{12}+P_{12}P_{22})y_2 \\ &= P_{11}y_1+P_{12}y_2=0, \end{aligned}$$

$$P_{22}x = P_{22}(P_{21}y_1+P_{22}y_2)+P_{21}(P_{11}y_1+P_{12}y_2)=x.$$

Then (4) shows that $x=0$. Before leaving the discussion of this theorem, let us avoid all misunderstanding about the relations (5) and (6) by noting explicitly that they mean this: for any x in $\mathfrak{H}$ the vectors $P_{11}x$ and $P_{12}x$ are both in $\mathfrak{D}(A)$, the domain of A, and $AP_{11}x=P_{21}x$, $AP_{12}x=P_{22}x$.

Corollary. *The operator A has a closed linear extension if and only if its characteristic matrix $<P_{ik}>$ verifies the condition* (4) *above.*

Proof. For A to have a closed linear extension it is necessary and sufficient that its graph be part of a closed subspace which is a graph. This condition is evidently equivalent to the stricter condition that the closed subspace generated by $\mathfrak{G}(A)$ be a graph, and hence to the condition asserted in the corollary.

As was first observed by von Neumann [1], an important role is played in the theory of non-bounded operators by the circumstances summarized as follows.

Theorem 3. *For an arbitrary operator A these propositions are equivalent:*

(*a*) *the domain $\mathfrak{D}(A)$ generates $\mathfrak{H}$;*

(*b*) *the adjoint operator A^* exists;*

(*c*) *the closed subspace* $\mathfrak{G}^*$ *of all vectors orthogonal to every vector in* $\mathfrak{G} = \mathfrak{G}(A)$ *is the inverted graph of an operator, necessarily the operator* $-A^*$;

(*d*) *the characteristic matrix of A verifies the condition*

$$P_{11}x = P_{21}x = 0 \qquad (7)$$

implies $x = 0$. *The operator* A^* *is necessarily a closed linear operator which can be described as the mapping*

$$A^*: P_{21}y_1 - (I - P_{22})\,y_2 \rightarrow (I - P_{11})\,y_1 - P_{12}y_2,$$

where y_1 *and* y_2 *are arbitrary vectors in* $\mathfrak{H}$. *In particular the identities*

$$I - P_{11} = A^*P_{21}; \qquad (8)$$

$$P_{12} = A^*(I - P_{22}) \qquad (9)$$

are verified.

Proof. We define ‡ a correspondence, not necessarily many-to-one, by putting $y \rightarrow y^*$ if and only if $(Ax)\,y = xy^*$ for every x in $\mathfrak{D}(A)$. Since the scalar product of the vectors (x_1, x_2) and (y_1, y_2) in $\mathfrak{H} \oplus \mathfrak{H}$ is $x_1y_1 + x_2y_2$, we see at once that this condition is equivalent to requiring that $(-y^*, y) \in \mathfrak{G}^*$. The correspondence just defined is therefore a many-to-one correspondence (by definition the adjoint operator A^*) if and only if $\mathfrak{D}(A)$ generates $\mathfrak{H}$; or, again, if and only if $\mathfrak{G}^*$ is the inverted graph of an operator (namely, $-A^*$). Since $I - P$ is the projection of $\mathfrak{H} \oplus \mathfrak{H}$ on $\mathfrak{G}^*$ we see by reasoning similar to that used in the proof of Theorem 2 that $\mathfrak{G}^*$ has this property if and only if (7) holds. In fact we note that the relations $(x, 0) \in \mathfrak{G}^*$ and $(x, 0) = (I - P)\,(x, 0) = (x - P_{11}x, P_{21}x)$ are equivalent; and hence that for $(x, 0) \in \mathfrak{G}^*$ to imply $x = 0$ it is necessary and sufficient that (7) hold. Thus the four propositions (a)–(d) are evidently equivalent. Since the vector $(I - P)\,(y_1, y_2) = ((I - P_{11})\,y_1 - P_{12}\,y_2, -P_{21}\,y_1 + (I - P_{22})\,y_2)$ varies over $\mathfrak{G}^*$ when y_1 and y_2 vary over $\mathfrak{H}$, the description of A^* as the mapping indicated in the theorem is evident. The relations (8) and (9) are obtained as special cases when we put $y_2 = 0$ and $y_1 = 0$ in turn.

‡ We write xy for the scalar product of the vectors x and y.

Corollary 1. *The operator A with domain $\mathfrak{D}(A)$ generating $\mathfrak{H}$ has a closed linear extension if and only if A^{**} exists. When A^{**} exists every closed linear extension of A is also an extension of A^{**}.*

Proof. The closed subspace generated by $\mathfrak{G}$ is identical with the set $\mathfrak{G}^{**}$ of all those vectors which are orthogonal to every vector in $\mathfrak{G}^{*}$. If A has a closed linear extension the graph of the latter contains $\mathfrak{G}$ and hence also $\mathfrak{G}^{**}$. Thus $\mathfrak{G}^{**}$ is a graph-necessarily the graph of the operator $(-(-A^{*})^{*}) = A^{**}$, in view of the theorem just proved. Conversely, if $\mathfrak{G}^{**}$ is a graph, as is the case if and only if A^{**} exists, then A^{**} is a closed linear extension of A; and every other closed linear extension of A, having a graph which contains $\mathfrak{G}^{**}$, is an extension of A^{**}.

Corollary 2. *When A is a closed linear operator, a necessary and sufficient condition for A^{*} to exist is that*

$$P_{11}x = 0 \text{ implies } x = 0 \text{—in other words, } P_{11} \tag{7'}$$

has an inverse.

Proof. In view of (5) we see that $P_{11}x = P_{21}x = 0$ if and only if $P_{11}x = 0$. Hence (7) now assumes the form (7').

These results show that it is useful to introduce the following definition.

Definition 2. *An operator A is said to be regular if it is closed linear and has a domain generating $\mathfrak{H}$.*

By combining the information we now have, it is easy to establish the following fundamental facts.

Theorem 4. *The adjoint of a regular operator A is regular, and $A^{**} = A$. The characteristic matrix of such an operator is expressible in terms of A by the formulas*

$$P_{11} = (I + A^{*}A)^{-1}, \quad P_{21} = A(I + A^{*}A)^{-1},$$
$$P_{12} = A^{*}(I + AA^{*})^{-1}, \quad P_{22} = AA^{*}(I + AA^{*})^{-1}, \tag{10}$$

where $I+A^*A$ *and* $I+AA^*$ *are self-adjoint operators inverse to* P_{11} *and* $I-P_{22}$ *respectively. The matrix* $<P_{ik}>$ *of a projection in* $\mathfrak{H}\oplus\mathfrak{H}$ *is the characteristic matrix for a regular operator if and only if the conditions* (4) *and* (7) *are both verified.*

PROOF. When A is closed and linear we have $\mathfrak{G}=\mathfrak{G}^{**}$; and when it is regular both $\mathfrak{G}^*$ and $\mathfrak{G}^{**}$ are graphs of operators—namely, the operators $-A^*$ and $A^{**}=A$ respectively. We can now apply the relations (5), (6), (8) and (9) to compute the characteristic matrix in the latter case. The operator $P_{21}=AP_{11}$ has as its range a part of the domain of A^* and $I-P_{11}=A^*P_{21}=A^*AP_{11}$. Consequently the operators A^*A and $I+A^*A$ have a common domain which contains the range of P_{11}. Evidently the identity $(I+A^*A)P_{11}=I$ is verified. Thus the operator P_{11}, which is self-adjoint in accordance with (2), has an inverse P_{11}^{-1} of which $I+A^*A$ is an extension. As is well known P_{11}^{-1} is also self-adjoint. Since the extension $I+A^*A$ is symmetric it must coincide with P_{11}^{-1}. This establishes the formula given in (10) for P_{11} and also yields that for P_{21} by an application of (5). In an entirely similar way we see that $P_{22}=AP_{12}=AA^*(I-P_{22})$ by virtue of (8) and (9), and hence that $I-P_{22}=(I+AA^*)^{-1}$ where $I+AA^*$ is a self-adjoint operator. The formula for P_{12} follows directly. That for P_{22} is evident also when it is recalled that the range of $(I+AA^*)^{-1}$ lies in the domain of AA^* and that $I=(I+AA^*)^{-1}+AA^*(I+AA^*)^{-1}$. The final statement of the theorem merely summarizes what we already know about the conditions (4) and (7).

COROLLARY. *For a regular operator* A, *the relations*

$$P_{12}\supset(I+A^*A)^{-1}A^*, \tag{11}$$

$$P_{21}\supset(I+AA^*)^{-1}A \tag{12}$$

are verified.

PROOF. We have $P_{12}=P_{21}^*=(AP_{11})^*\supset P_{11}A^*$ and $P_{21}=P_{12}^*=(A^*(I-P_{22}))^*\supset(I-P_{22})A^{**}=(I-P_{22})A$.

We turn now to a consideration of the commutation properties of operators. First let us recall the fundamental definition on which the discussion is based.

Definition 3. *An arbitrary operator A and a bounded linear operator B are said to commute (one with the other) if $BA \subset AB$—that is, if B maps $\mathfrak{D}(A)$ into itself and $ABx = BAx$ for every vector in $\mathfrak{D}(A)$.*

We can immediately prove the following basic results.

THEOREM 5. *If a bounded linear operator B and its adjoint B^* both commute with A, then both also commute with all four components of the characteristic matrix of A. For a closed linear operator A a strengthened form of the converse is true: if either B or B^* commutes with these components, then both operators commute with A.*

PROOF. Let $S = S(B)$ be the operator defined by putting $S_{11} = S_{22} = B$ and $S_{12} = S_{21} = O$. It is easy then to see that B commutes with A if and only if S maps $\mathfrak{G} = \mathfrak{G}(A)$ into itself: for the relations $(x_1, x_2) \in \mathfrak{G}$ and $S(x_1, x_2) = (Bx_1, Bx_2) \in \mathfrak{G}$ are respectively equivalent to the relations $x_2 = Ax_1$ and $Bx_2 = ABx_1$; and the first therefore implies the second if and only if $BA \subset AB$. Whenever S maps $\mathfrak{G}$ into itself, it also maps into itself the closed subspace generated by $\mathfrak{G}$. The latter situation is realized if and only if $PSP = SP$ where $P = P(A)$ is the projection of $\mathfrak{H} \oplus \mathfrak{H}$ on the subspace in question. Accordingly the condition $PSP = SP$ is necessary for B to commute with A; and when A is closed and linear, so that its graph coincides with the range of P, the condition becomes a sufficient one as well. Since $S(B^*) = S(B)^* = S^*$, a like connection obtains between the commutativity of B^* with A and the condition $PS^*P = S^*P$. Whenever S and P commute, we have $PSP = SP^2 = SP$ and also $S^*P = S^*P^* = (PS)^* = (SP)^* = P^*S^* = PS^*$. Similarly the commutativity of S^* and P implies that of S and P, along with the relation $PS^*P = S^*P$. On the other hand if

$PSP = SP$ and $PS^*P = S^*P$ we see that $PS = P^*S^{**} = (S^*P)^* = (PS^*P)^* = P^*S^{**}P^* = PSP = SP$. In other words, $PSP = SP$ and $PS^*P = S^*P$ if and only if S and P commute or, equivalently, S^* and P commute. The proof of the theorem is now completed by observing that $S = S(B)$ and P commute if and only if B commutes with the components of the matrix $< P_{ik} >$. It would be easy to write down the corresponding conditions for the validity of the relation $PSP = SP$, but they will not be needed in the sequel.

We recall that a closed subspace $\mathfrak{M}$ is said to *reduce* an operator A if its projection commutes with A—in other words, if $\mathfrak{D}(A)$ is projected onto $\mathfrak{D}(A) \cap \mathfrak{M}$ and the restriction of A to the latter set is an operator in $\mathfrak{M}$. For $\mathfrak{M}$ to reduce A, it is necessary and sufficient that $\mathfrak{M}^*$ reduce A. When A is bounded and linear, the following assertions are equivalent: $\mathfrak{M}$ reduces A, $\mathfrak{M}^*$ reduces A, A maps each of the closed subspaces $\mathfrak{M}$ and $\mathfrak{M}^*$ into itself, A and A^* both map $\mathfrak{M}$ into itself. In the present connection we note two corollaries to Theorem 5.

COROLLARY 1. *If the closed subspace $\mathfrak{M}$ reduces an operator A then it also reduces the components of the characteristic matrix for A; and, when A is a closed linear operator, the converse is also true.*

COROLLARY 2. *If the closed subspace $\mathfrak{M}$ reduces an operator A, then the characteristic matrix for the operator defined in $\mathfrak{M}$ by restricting A to the set $\mathfrak{D}(A) \cap \mathfrak{M}$ is obtained by restricting to $\mathfrak{M}$ the components of the characteristic matrix for A in $\mathfrak{H}$.*

These commutation properties will now be applied to the study of normal operators, introduced in the following definition.

Definition 4. *An operator A is said to be normal if it is regular and $AA^* = A^*A$.*

It should be noted that when A is bounded linear this definition means merely that A and its adjoint

commute; but that when A is a non-bounded operator there is no *a priori* reason for supposing that either of the operators A^*A and AA^* has a big enough domain to give the definition substantial comment. However, Theorem 5 shows that the hypothesis of regularity guarantees each of these operators to have a fairly large domain. Indeed it actually yields the following result, which is quite as strong as could be asked.

THEOREM 6. *For a regular operator A these assertions are equivalent*:

(*a*) *A is normal*; (*b*) *A^* is normal*;

(*c*) $A^*A \subset AA^*$; (*d*) $AA^* \subset A^*A$;

(*e*) *the components of the characteristic matrix for A commute with one another.*

PROOF. When A is normal, its adjoint A^* is regular by Theorem 4, and $A^{**} = A$. Hence $A^* A^{**} = A^{**} A^*$ and A^* is normal. Conversely, if A^* is normal, then $A = A^{**}$ and $A^* A^{**} = A^{**} A^*$ imply that A is normal Evidently, (*c*) and (*d*) are trivial consequences of (*a*). On the other hand, (*c*) implies that $I - P_{22} = (I + AA^*)^{-1}$ is an extension of $P_{11} = (I + A^*A)^{-1}$. Since both these operators are defined throughout $\mathfrak{H}$, they must be identical. It follows that AA^* coincides with A^*A and that A is normal. In a similar way, (*d*) also implies (*a*). If A is normal, we see as before that $I - P_{22} = P_{11}$. Hence (5) and (12) yield $AP_{11} = P_{21} \supset P_{11} A$. Since the self-adjoint operator P_{11} thus commutes with A, we infer from Theorem 5 that it also commutes with every component of the matrix $< P_{ik} >$. Clearly the same assertion can be made concerning P_{22}. To derive (*e*) from (*a*) we still have to show that P_{12} and P_{21} commute. This is easy to do with the use of (3) since $P_{12} P_{21} = P_{11} - P_{11}^2 = (I - P_{22}) - (I - P_{22})^2 = P_{22} - P_{22}^2 = P_{21} P_{12}$. Conversely, when (*e*) holds we see that every component of the characteristic matrix commutes with A. It is also easily verified that P_{ik} commutes with A^*: indeed

if $y \epsilon \mathfrak{D}(A^*)$ we have for every x in $\mathfrak{D}(A)$ the relations $(Ax)(P_{ik}y) = (P_{ik}^* Ax)y = (P_{ki}Ax)y = (AP_{ki}x)y = (P_{ki}x)(A^*y) = x(P_{ik}A^*y)$, which imply that $P_{ik}y \epsilon \mathfrak{D}(A^*)$ and $A^*P_{ik}y = P_{ik}A^*y$. Accordingly we have $(I-P_{22})P_{11}^{-1} = (I-P_{22}).(I+A^*A) = (I-P_{22}) + (I-P_{22})A^*A = (I-P_{22}) + A^*(I-P_{22})A = (I-P_{22}) + P_{12}A \subset (I-P_{22}) + AP_{12} = (I-P_{22}) + P_{22} = I$. Thus $I-P_{22}$ coincides with P_{11} on the range of P_{11}^{-1}—that is, on $\mathfrak{D}(A^*A)$. Since this set generates $\mathfrak{H}$ and the operators $I-P_{22}$ and P_{11} are continuous, they must be identical. It then follows as before that $A^*A = AA^*$ and that A is normal.

In order to establish another criterion for normality, due originally to von Neumann [1], we first prove a useful lemma.

Lemma 1. *If A is normal operator and B is the restriction of A to the set $\mathfrak{D}(A^*A) = \mathfrak{D}(AA^*)$, then $\mathfrak{G}(A)$ is the closure of $\mathfrak{G}(B)$ and $A^* = B^*$, $A = B^{**}$.*

Proof. We begin by showing that $A^* = B^*$. Let y and y^* be vectors such that $y^* = By$; or, equivalently, such that $(Bx)y = xy^*$ for every x in $\mathfrak{D}(B) = \mathfrak{D}(A^*A) = \mathfrak{D}(AA^*)$. Putting $x = P_{11}z$, $z \epsilon \mathfrak{H}$, we see that $(BP_{11}z)y = (P_{11}z)y^*$ for every vector z. Since $BP_{11} = AP_{11} = P_{21} = P_{12}^*$ we conclude that $z(P_{12}y) = z(P_{11}y^*)$ for every z and hence $P_{12}y = P_{11}y^*$. Accordingly $(I-P_{11})y = P_{22}y = AP_{12}y = AP_{11}y^*$. Thus $y = P_{11}y + AP_{11}y^*$ is seen to be in the domain of A^* and $A^*y = A^*P_{11}y + A^*AP_{11}y^* = P_{12}y + y^* - P_{11}y^* = y^*$. It follows that $A^* \supset B^*$. Since $B \subset A$ implies $B^* \supset A^*$ we must therefore have $A^* = B^*$. Thus $A = A^{**} = B^{**}$ as well; and from this relation we see that the closure of the graph $\mathfrak{G}(B)$ is $\mathfrak{G}(B)^{**} = \mathfrak{G}(B^{**}) = \mathfrak{G}(A)$.

We can now prove von Neumann's criterion.

Theorem 7. *A regular operator A is normal if and only if A and A^* have a common domain and $|Ax| = |A^*x|$ for every x in $\mathfrak{D}(A) = \mathfrak{D}(A^*)$.*

Proof. First, let us suppose that A is normal. Then by the lemma just proved we can associate with any vector (x, Ax) in $\mathfrak{G}(A)$ a sequence of vectors (x_n, Bx_n) in $\mathfrak{G}(B)$ which converges to it in $\mathfrak{H} \oplus \mathfrak{H}$. Thus in $\mathfrak{H}$ the sequences $\{x_n\}$ and $\{Ax_n\} = \{Bx_n\}$. converge to x and to Ax respectively. Since $x_n \epsilon \mathfrak{D}(A^*A) = \mathfrak{D}(AA^*)$ we evidently have $|Ax_n|^2 = (Ax_n)\ (Ax_n) = x_n\ (A^*Ax_n) = x_n\ (AA^*x_n) = (A^*x_n)\ (A^*x_n) = |A^*x_n|^2$ and hence $|Ax_n| = |A^*x_n|$. Similarly $|Ax_m - Ax_n| = |A^*x_m - A^*x_n|$. The latter equality shows that $\{A^*x_n\}$ is a Cauchy sequence. As $\mathfrak{H}$ is complete $\{A^*x_n\}$ has a limit y^*. Clearly $|y^*| = |Ax|$, by virtue of the first equality. Since A^* is a closed linear operator, and since the sequences $\{x_n\}$ and $\{A^* x_n\}$ have the limits x and y^* respectively it follows that $A^*x = y^*$. Thus the domain of A^* contains that of A and $|Ax| = |A^*x|$ for every x in $\mathfrak{D}(A)$. By symmetry it is clear that the domain of A^* is contained in that of $A = A^{**}$, so that in fact $\mathfrak{D}(A^*) = \mathfrak{D}(A)$. On the other hand, let A be a regular operator with $|Ax| = A^*x|$ whenever $x \epsilon \mathfrak{D}(A) = \mathfrak{D}(A^*)$. We consider the set $\mathfrak{M} = (x;\ |Ax| = |A^*x| = 0)$. The ranges of both A and A^* have $\mathfrak{M}$ as their orthogonal complement: for $(Ax)y = 0$ for every x in $\mathfrak{D}(A)$ if and only if $A^*y = 0$, $(A^*x)y = 0$ for every x in $\mathfrak{D}(A^*)$ if and only if $Ay = A^{**}y = 0$, and our hypothesis implies that $Ay = 0$ and $A^*y = 0$ are equivalent relations. Thus if $\mathfrak{N} = \mathfrak{M}^*$ is the orthogonal complement of $\mathfrak{M}$, the ranges of both operators A and A^* generate $\mathfrak{N}$. The sets $\mathfrak{M}$ and $\mathfrak{N}$ are closed subspaces of $\mathfrak{H}$ such that $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{N}$. The correspondence $Ax \to A^*x$ clearly defines a one-to-one linear and isometric mapping of the range of A onto that of A^*. This mapping can be extended by continuity into a unique unitary mapping of $\mathfrak{N}$ onto $\mathfrak{N}$. The unique linear mapping defined on $\mathfrak{H}$ so as to coincide with this one in $\mathfrak{N}$ and with the identity mapping in $\mathfrak{M}$ is a unitary operator U. By definition we evidently have $UA = A^*$.

On the other hand, we have $A^*U^* = A$, as we shall now show. If x is in $\mathfrak{D}(A)$, then for any y in $\mathfrak{D}(A)$ we have $(Ay)(U^*x) = (UAy)x = (A^*y)x = yAx$, and we conclude that $U^*x \in \mathfrak{D}(A^*)$ and $A^*U^*x = Ax$. Similarly, if U^*x is in $\mathfrak{D}(A^*)$, then for any y in $\mathfrak{D}(A)$ we have $y(A^*U^*x) = (Ay)(U^*x) = (UAy)x = (A^*y)x$, and we conclude that $x \in \mathfrak{D}(A)$ and $Ax = A^{**}x = A^*U^*x$. Thus the desired identity is proved. Now it is evident that $A^*A = A^*IA = A^*(U^*U)A = (A^*U^*)(UA) = AA^*$ and hence that A is normal.

Corollary 1. *A symmetric operator A is normal if and only if it is self-adjoint, $A^* = A$.*

Proof. Let A be normal and symmetric. We then have $\mathfrak{D}(A) = \mathfrak{D}(A^*)$ and $A \subset A^*$. Thus $A^* = A$. On the other hand $A = A^*$ implies that A is a regular operator satisfying the condition $A^*A = AA^*$ for normality.

Using a convenient and self-explanatory terminology without making formal definitions we have an analogous result.

Corollary 2. *An anti-symmetric operator A is normal if and only if is anti-adjoint, $A^* = -A$.*

Proof. We have $\mathfrak{D}(A) = \mathfrak{D}(A^*)$ and $-A \subset A$ when A is anti-symmetric and normal; and we conclude that $A^* = -A$. We have $\mathfrak{D}(A) = \mathfrak{D}(A^*)$ and $A^*A = AA^*$ when $A^* = -A$.

A useful extension of part of Theorem 6 is contained in our next theorem.

Theorem 8. *In order that the components of the characteristic matrix for a closed linear operator A commute with one another it is necessary and sufficient that A be normal in a closed subspace of $\mathfrak{H}$.*

Proof. Let $\mathfrak{N}$ be the closed subspace of all those vectors x such that $P_{11}x = 0$; and let $\mathfrak{M} = \mathfrak{N}^*$ be its orthogonal complement. If the components of $< P_{ik} >$ commute with one onother, we can easily verify that each

of them maps $\mathfrak{N}$ onto (o). For P_{21} this follows from the identity (5). Then $P_{21}^* = P_{12}$ commutes with P_{21}, so that both P_{12} and P_{21} are normal and $|P_{12}x| = |P_{21}x|$ in accordance with Theorem 7. It then follows that $x \epsilon \mathfrak{N}$ implies $P_{12}x = 0$. Finally when $x \epsilon \mathfrak{N}$ we have for $P_{22}x$ the relations $P_{12}(P_{22}x) = P_{22}(P_{12}x) = 0$ and $(I - P_{22}) P_{22} x$ $P_{22}x - P_{22}^2 x = P_{21} P_{12} x = 0$. By (4) we conclude that $P_{22} x = 0$. Thus $\mathfrak{N}$ reduces every matrix-component P_{ik}; and so also does $\mathfrak{M}$. It follows that $\mathfrak{M}$ and $\mathfrak{N}$ reduce A, in accordance with Theorem 5, Corollary 1; and that the characteristic matrix for the restriction A to $\mathfrak{D}(A) \cap \mathfrak{N}$ is obtained, as stated in Corollary 2 to the same theorem, by restricting the components of $< P_{ik} >$ to $\mathfrak{N}$. The result of this operation is clearly the matrix with every component equal to the operator O in $\mathfrak{N}$. It follows that the graph of the restricted operator consists of the vector (o, o) alone. Thus $\mathfrak{D}(A) \cap \mathfrak{N} = (0)$. Hence A is a closed linear operator in the closed subspace $\mathfrak{M}$. It is regular in $\mathfrak{M}$ because the restriction of P_{11} to $\mathfrak{M}$ has the property (7): if $x \epsilon \mathfrak{M}$ and $P_{11} x = 0$, then $x \epsilon \mathfrak{M} \cap \mathfrak{N}$ and $x = 0$. Since the restrictions of the operators P_{ik} to $\mathfrak{M}$ commute with one another, Theorem 6 and Corollary 2 to Theorem 5 show that A is normal in $\mathfrak{M}$. On the other hand, if A is normal in a closed subspace $\mathfrak{M}$ of $\mathfrak{H}$, then $\mathfrak{M}$ and its orthogonal complement $\mathfrak{N}$ both reduce A and hence also the components of $< P_{ik} >$. Thus the components in question commute if and only if their restrictions to $\mathfrak{M}$ and to $\mathfrak{N}$ respectively commute as operators in those subspaces. Now the desired commutation property holds in $\mathfrak{M}$ because A is normal in $\mathfrak{M}$; and the analogous property holds in $\mathfrak{N}$ because $\mathfrak{D}(A) \cap \mathfrak{N} = (0)$ so that the restriction of A to $\mathfrak{N}$ has graph consisting of the vector (o, o) alone and characteristic matrix with every component equal to the operator O in $\mathfrak{N}$.

For the purposes of spectral theory (which we shall not discuss here) it is convenient to note the following

version of some of the preceding results concerning normal operators.

THEOREM 9. *A two-rowed matrix of bounded linear operators in* $\mathfrak{H}$ *is the characteristic matrix of a normal operator A if and only if it has the form* $\begin{pmatrix} B & C \\ C^* & I-B \end{pmatrix}$, *where B is an invertible self-adjoint operator, and C is a normal operator which commutes with B and satisfies the identity* $CC^* = B - B^2$. *In terms of this matrix, A and* A^* *are given by the identities* $A = B^{-1}C$, $A^* = B^{-1}C^*$.

PROOF. If A is a normal operator its characteristic matrix clearly has the above form with $B = P_{11}$ and $C = P_{21}$. The commutation relations for B, C and C^* follow from Theorem 6, and the identity $CC^* = B - B^2$ from (3). The fact that the relations $y = Ax$, $(x, y) \in \mathfrak{G}(A)$, $(x, y) = (Bx + C^*y, \ Cx + (I-B)y)$, and $By - Cx = C^*y - (I-B)x = 0$ are equivalent, allows us to show without any difficulty that the relations $y = Ax$ and $By = Cx$ are equivalent, i.e. that $A = B^{-1}C$. Indeed, it is enough to show that $By = Cx$ implies $C^*y - (I-B)x = 0$, in accordance with the relations $C^*y - (I-B)x = B^{-1}(BC^*y - (B-B^2)x) = B^{-1}(C^*By - C^*Cx) = B^{-1}C^*(By - Cx) = 0$. Similarly the fact that relations $z = A^*x$, $(-z, x) \in \mathfrak{G}(A)^*$, $(-z, x) = (-(I-B)z - C^*x, \ Cz + Bx)$ and $Bz - C^*x = Cy - (I-B)x = 0$ are equivalent, allows us to conclude that $A^* = B^{-1}C^*$: for if $Bz = C^*x$ we have also

$Cz - (I - Bx) = B^{-1}(CBz - (B-B^2)x) = B^{-1}C(Bz - C^*x) = 0$.

On the other hand, let a matrix of the indicated form be given. Then it is easy to verify the conditions (2) and (3) by direct computation. To show that the matrix is the characteristic matrix of a closed linear operator A it therefore suffices to verify (4); but this is trivial since $Bx = 0$ implies $x = 0$ by hypothesis. At the same time Corollary 2 of Theorem 3 shows that the operator A is regular. Theorem 6(e) shows that A is even normal, completing the proof.

COROLLARY. *A is self-adjoint in Theorem 9, if and only if C is self-adjoint.*

By way of illustrating the connections between operators and their characteristic matrices, we conclude this section with a brief list of the properties which distinguish the matrices belonging to operators of various important special types.

THEOREM 10. *Let A be a regular operator. Then*

(*a*) *A is bounded if and only if $P_{11} \geqslant \gamma I$ for some $\gamma > 0$;*

(*b*) *A is self-adjoint if and only if the components of its characteristic matrix are mutually commuting self-adjoint operators;*

(*c*) *A is a projection if and only if $P_{12} = P_{21} = P_{22} = I - P_{11}$;*

(*d*) *A is unitary if and only if $P_{11} = P_{22} = \frac{1}{2} I$.*

PROOF. In considering (*a*), let us first suppose that A is bounded, with bound α. Then $P_{11}^{-1} = I + A^*A$ is also bounded. Its bound can in fact, be computed as $\beta = 1 + \alpha^2$. For any vector x we know that $y = P_{11}x$ is in the domain of P_{11}^{-1}. Hence $|x| = |P_{11}^{-1}y| \leqslant \beta |y|$ and $|P_{11}x| \geqslant \gamma |x|$, $\gamma = 1/\beta$. Since P_{11} is self-adjoint, it follows that $0 \leqslant |(P_{11} - \gamma I)x|^2 = |P_{11}x|^2 - 2\gamma((P_{11}x)x) + \gamma^2 |x|^2$ and hence that $2\gamma((P_{11}x)x) \geqslant |P_{11}x|^2 + \gamma^2 |x|^2 \geqslant 2\gamma^2 |x|^2$ or $((P_{11} - \gamma I)x)x = (P_{11}x)x - \gamma |x|^2 \geqslant 0$. Thus $P_{11} \geqslant \gamma I$. When the latter relation holds for some $\gamma > 0$ we put $P_{11}x = y$ as before and apply Schwarz's inequality to obtain the relations $\gamma |P_{11}^{-1}y|^2 \leqslant y(P_{11}^{-1}y) \leqslant |y| \, |P_{11}^{-1}y|$ and $|P_{11}^{-1}y| \leqslant \beta |y|$ where $\beta = 1/\gamma$. If γ is the greatest lower bound for P_{11}, then β is evidently the bound of P_{11}^{-1}. It is then easily seen that A^*A has $\beta - 1$ as its bound, and that in turn A has $\alpha = (\beta - 1)^{1/2}$ as its bound. Thus (*a*) is established. When A belongs to any of the three other special types, it is normal and has a characteristic matrix of the form given in Theorem 9. Since the relations $A^* = A$, $B^{-1}C^* = B^{-1}C$, and $C^* = C$ are then equivalent, we conclude that A is self-adjoint if and only if $C^* = C$. Furthermore, we can add that A is a projection if and only if the further condition $B + C = I$ is verified, in which

case necessarily $C = \frac{1}{2}A$, $B = I - \frac{1}{2}A$. Indeed if A is a projection we can put $A^* = A^*A = AA^* = A$ in (10), obtaining $C = C^* = I - B$; and if $B + C = I$ we have $AB = B^{-1}CB = B^{-1}BC = C$, $AC = B^{-1}C^2 = B^{-1}(B - B^2) = I - B = C$, $A = AI = A(B+C) = AB + AC = 2C$, and $A^2 = 2AC = 2C = A$, so that A is a projection. When A is unitary, we can put $A^*A = AA^* = I$ in (10), obtaining $B = \frac{1}{2}I$ and $C = \frac{1}{2}A$; and if $B = \frac{1}{2}I$ then $A = B^{-1}C = 2C$ and $A^*A = AA^* = 4CC^* = 4B - 4B^2 = I$, so that A is unitary. Thus the conditions stated in (*b*), (*c*), (*d*) as necessary for A to be respectively self-adjoint, a projection, or unitary are clearly justified. Suppose now that the characteristic matrix of a regular operator A satisfies the condition stated as necessary in (*b*). According to Theorem 6, the operator A is normal; and, as we saw above, it is self-adjoint. Again, suppose that a matrix $< P_{ik} >$ is given satisfying the condition $P_{12} = P_{21} = P_{22} = I - P_{22}$ of (*c*). If we put $B = P_{11}$ and $C = P_{21}$, we have $B + C = I$. Imposing the conditions (2) and (3), which are necessary if $< P_{ik} >$ is to be a characteristic matrix, we find that they reduce to the requirement that B and $C = I - B$ be self-adjoint operators satisfying the relation $CC^* = B - B^2$. Thus if conditions (4) and (7) can be verified, we may conclude that the given matrix is the characteristic matrix of a normal operator A in accordance with Theorems 6 and 9; and then, as we have already seen above, A must be a projection and $C = \frac{1}{2}A$. Now in fact the conditions (4) and (7) both reduce here to the requirement that $Bx = Cx = 0$ imply $x = 0$; but since $x = (B+C)x = Bx + Cx$ this requirement is automatically satisfied. Finally, let us suppose that a matrix $< P_{ik} >$ satisfying the condition $P_{11} = P_{22} = \frac{1}{2}I$ of (*d*) is given. If we put $B = P_{11} = \frac{1}{2}I$ and $C = P_{21}$ and impose the conditions (2) and (3) we see that the matrix $< P_{ik} >$ assumes the form indicated in Theorem 9: for B is trivially an invertible self-adjoint operator and, by direct substitution in (3), $CC^* = B - B^2 = \frac{1}{4}I$ and $C^*C = B - B^2 = \frac{1}{4}I$. Hence

$<P_{ik}>$ is the characteristic matrix of a normal operator A, which, as we have already seen above, must be unitary and identical with $2C$.

Some of the results noted in the proof of Theorem 10 can conveniently be stated as corollaries.

COROLLARY 1. *A matrix is the characteristic matrix of a projection A if and only if it has the form*

$$\begin{pmatrix} I-\frac{1}{2}E & \frac{1}{2}E \\ \frac{1}{2}E & \frac{1}{2}E \end{pmatrix},$$

where E is a projection, necessarily identical with A.

COROLLARY 2. *A matrix is the characteristic matrix of a unitary operator A if and only if it has the form*

$$\begin{pmatrix} \frac{1}{2}I & \frac{1}{2}U \\ \frac{1}{2}U^* & \frac{1}{2}I \end{pmatrix},$$

where U is a unitary operator, necessarily identical with A.

A similar result, which includes Corollary 2 just above as a special case, is of sufficient interest to be noted here.

THEOREM 11. *A matrix is the characteristic matrix of a closed linear isometric operator A if and only if it has the form*

$$\begin{pmatrix} \frac{1}{2}U^*U & \frac{1}{2}U \\ \frac{1}{2}U^* & \frac{1}{2}UU^* \end{pmatrix},$$

where $UU^*U = U$. Here it is necessary that U be that extension of A which maps $\mathfrak{D}(A)^*$ onto (0), U^* that extension of A^{-1} which maps $\mathfrak{D}(A^{-1})^*$ onto (0), U^*U the projection of $\mathfrak{H}$ on $\mathfrak{D}(A)$, and UU^* the projection of $\mathfrak{H}$ on $\mathfrak{D}(A^{-1})$.

PROOF. Let $<P_{ik}>$ be the characteristic matrix of a closed linear operator A. Then the description of A given in Theorem 2 shows that A is isometric if and only if $|P_{11}y_1+P_{12}y_2| = |P_{21}y_1+P_{22}y_2|$ for all y_1 and y_2. This condition can be put in the form $((P_{11}^2-P_{12}P_{21})y_1, y_1)$ $+2\mathbf{R}((P_{21}P_{11}-P_{22}P_{21})y_1, y_2)+((P_{21}P_{12}-P_{22}^2)y_2, y_2) = 0$, by application of (2). Putting $y_2 = 0$ and $y_1 = 0$ in turn, we see that this relation is equivalent to the separate vanishing of the three summands. Since the operators $P_{11}^2-P_{12}P_{21}$ and $P_{21}P_{12}-P_{22}^2$ are self-adjoint the terms containing

them vanish if and only if these operators coincide. and the remaining term vanishes if and only if $P_{21} P_{11} - P_{22} P_{21} = 0$ also, as we see by choosing y_2 to be equal to the first factor in the scalar product involved. These identities form with (2) and (3) a set of relations equivalent to the set $P_{11} = 2P_{11}^2 = 2P_{12} P_{21}$, $P_{22} = 2P_{22}^2 = 2P_{21} P_{12}$, $P_{21} = 2P_{21} P_{11} = 2P_{22} P_{21}$. If we change our notation by writing $2P_{21} = U$, we have $U^* = 2P_{12}$ by (2); and the last equations reduce to the equivalent relations $2P_{11} = U^*U$, $2P_{22} = UU^*$, $U = UU^*U$. Moreover if U is an arbitrary bounded linear operator such that $UU^*U = U$, we may define P_{ik} by the preceding equations and verify directly that the conditions (2), (3), (4) are satisfied. Thus $< P_{ik} >$ is the characteristic matrix of a closed linear operator A, necessarily isometric in accordance with the discussion above. The self-adjoint operators $E = 2P_{11} = U^*U$, $F = 2P_{22} = UU^*$ are idempotent and are therefore projections. By (5) we have $U = AE$, so that $\mathfrak{D}(A)$ contains the range of E. On the other hand, $x \in \mathfrak{D}(A)$ implies that x has the expression $x = P_{11} y_1 + P_{12} y_2 = \frac{1}{2} Ey_1 + \frac{1}{2} U^* y_2$ and hence $x = Ex$, since $EU^* = (UE)^* = (UU^*U)^* = U^*$. Thus $x = Ex$ if and only if $x \in \mathfrak{D}(A)$—in other words, E is the projection of $\mathfrak{H}$ on $\mathfrak{D}(A)$; and U is that linear extension of A which carries $\mathfrak{D}(A)^*$ onto (0). The range of U is identical with that of A and hence with $\mathfrak{D}(A^{-1})$. Clearly $x \in \mathfrak{D}(A)$ implies $x = Ex = U^*Ux$, so that U^* is an extension of A^{-1}. Since $F = UU^*$, the range of F is part of that of U and hence is part of $\mathfrak{D}(A^{-1})$. On the other hand $Fx = 0$ implies $U^*x = EU^*x = U^*Fx = 0$ so that the range of F cannot be a proper part of $\mathfrak{D}(A^{-1})$. Thus F is the projection of $\mathfrak{H}$ on $\mathfrak{D}(A^{-1})$, and $U^* = A^{-1}F$. This completes the proof.

3. Commutants and monogenic subspaces.

From a geometrical point of view the central problem in the theory of linear operators is to determine the closed subspaces which reduce a given family or set of operators.

The equivalent algebraic problem is to determine all the projections which commute with every one of the given operators. It is then only natural to consider the apparently broader problem of finding all the bounded linear operators which, simultaneously with their adjoints, commute with every one of the given operators. The spectral theory shows in the complex case that the operators sought can all be generated by the projections which fulfil the conditions of the previous problem. Thus the two problems are, in fact, equivalent here. These considerations suggest the importance of the following definition.

Definition 5. *If* $\mathbf{A}$ *is any set of operators, the set* $\mathbf{A}'$ *of all those bounded linear operators which, together with their adjoints, commute with every operator in* $\mathbf{A}$ *is called the commutant of* $\mathbf{A}$.

Here we shall review briefly some of the elementary theory of commutants, developed originally by von Neumann [3], since the case where $\mathbf{A}$ contains operators which are not linear or not closed and the case where $\mathfrak{H}$ admits the quaternions as scalars, both require some comment.

In the general case, we have first the following result.

THEOREM 12. *The commutant* $\mathbf{A}'$ *is a multiplicative self-adjoint set which contains the operator* I, *and is contained in the commutant* $(\mathbf{A}^*)'$ *of the set* $\mathbf{A}^*$ *consisting of the adjoints* $A^*, A \in \mathbf{A}$. *In general* αI, $\alpha \neq 1$, *is not in* $\mathbf{A}'$: *for in all cases,* $\alpha I \in \mathbf{A}'$ *and* $x \in \mathfrak{D}(A)$, $A \in \mathbf{A}$ *imply* $A(\alpha x) = \alpha Ax$; *and in the quaternionic case,* αI *is linear if and only if* α *is a real quaternion.*

PROOF. By definition, $\mathbf{A}'$ is self-adjoint in the sense that when it contains B it also contains B^*. If B and C are in $\mathbf{A}'$, for any $A \in \mathbf{A}$ any $x \in \mathfrak{D}(A)$ we have $Cx \in \mathfrak{D}(A)$, $(BC)x = B(Cx) \in \mathfrak{D}(A)$, $A(BCx) = (AB)(Cx) = B(ACx) = B(CAx) = (BC)(Ax)$ and conclude that BC commutes with A; and at the same time we see that $(BC)^* = C^*B^*$ commutes with A. Thus $\mathbf{A}'$ contains BC whenever it

contains B and C. In general no operator in $\mathbf{A}$ need have a domain which generates $\mathfrak{H}$, and the set $\mathbf{A}^*$ may thus be empty. However, if $A \in \mathbf{A}$ and A^* exists, we can show that whenever B and B^* commute with A they also commute with A^*. For, if $x \in \mathfrak{D}(A)^*$ then for any y in $\mathfrak{D}(A)$ we have $(Bx)(Ay) = x(B^*Ay) = x(AB^*y) = (A^*x)(B^*y) = (BA^*x)\, y$ and thus infer that $Bx \in \mathfrak{D}(A^*)$ and $A^*Bx = BA^*x$. Accordingly $\mathbf{A}' \subset (\mathbf{A}^*)'$. The remaining assertions of the theorem are evident.

Under special conditions on the set $\mathbf{A}$ more can be said about $\mathbf{A}'$. We collect a number of useful observations on this point as our next theorem.

THEOREM 13. *If $\mathbf{A}$ consists of linear operators, then $\mathbf{A}'$ is a self-adjoint linear algebra over the center of the scalar system of $\mathfrak{H}$. If $\mathbf{A}$ consists of closed operators, then $\mathbf{A}'$ contains every operator B such that B and B^* are in the closure of $\mathbf{A}'$ in the strong operator-topology. If $\mathbf{A}$ consists of closed linear operators, then there exists a set $\mathbf{B}$ of bounded linear operators such that $\mathbf{A}' = \mathbf{B}'$. If $\mathbf{A}$ consists of regular operators, then $\mathbf{A}'$ is closed in the weak operator-topology (hence also in the strong); and $\mathbf{A}' = (\mathbf{A}^*)'$.*

PROOF. If the operators in $\mathbf{A}$ are all linear, it is easy to verify that whenever $B \in \mathbf{A}'$ and $C \in \mathbf{A}'$ then $B + C \in \mathbf{A}'$ and $\alpha B \in \mathbf{A}'$, provided that α is in the center of the scalar system for $\mathfrak{H}$, i.e. is restricted to be a real quaternion in the quaternionic case. If $\mathbf{A}$ consists of closed linear operators and if B is in the strong closure of $\mathbf{A}'$, we can find a sequence of operators $C_n \in \mathbf{A}'$ such that $|Bx - C_n x| \leq 1/n$ and $|BAx - C_n Ax| \leq 1/n$ for any fixed vector x in $\mathfrak{D}(A)$. Hence we see that $\{C_n x\}$ and $\{A(C_n x)\} = \{C_n Ax\}$ are sequences which converge respectively to Bx and to BAx. Since A is closed, we conclude that $Bx \in \mathfrak{D}(A)$ and that $ABx = BAx$. Thus B commutes with every A in $\mathbf{A}$. The fact that the formation of adjoints is not a continuous operation with respect to the strong operator-topology

prevents our deriving any conclusion about B^*. By assuming that both B and B^* are in the strong closure of $\mathbf{A}'$, we are able to conclude that $B \varepsilon \mathbf{A}'$, as asserted in the theorem. When the operators in $\mathbf{A}$ are all closed and linear, we can choose the set $\mathbf{B}$ as the set of all the components of the characteristic matrices for the operators $A \varepsilon \mathbf{A}$. By Theorem 5, we see that $\mathbf{A}' = \mathbf{B}'$. Finally, if $\mathbf{A}$ consists of regular operators and if B is in the closure of $\mathbf{A}'$ in the weak operator-topology, we can find a sequence of operators $C_n \varepsilon \mathbf{A}'$ such that

$$|((B-C_n)x)A^* y| \leqslant 1/n, \quad |((B-C_n)Ax)y| \leqslant 1/n,$$
$$|x((B-C_n)A^* y)| \leqslant 1/n, \quad |((Bx)(B \quad C_n))y| \leqslant 1/n,$$

for any fixed x and y in the domains of A and of A^* respectively. Noting that $(C_n x)A^* y = (AC_n x)y = (C_n Ax)y$ and $x(C_n A^* y) = x(A^* C_n y) = (Ax)(C_n y)$ we conclude that $|(Bx)(A^* y) - (BAx)y| \leqslant 2/n$, $|x(BA^* y) - (Ax)(By)| \leqslant 2/n$, for $n = 1, 2, 3, \ldots$. Accordingly we have $(Bx)(A^* y) = (BAx)y$ and $(B^* x)(A^* y) = x(BA^* y) = (Ax)(By) = (B^* Ax)y$ for every $x \varepsilon \mathfrak{D}(A)$ and $y \varepsilon \mathfrak{D}(A^*)$. Since $A = A^{**}$ by hypothesis, we now see that $Bx \varepsilon \mathfrak{D}(A)$, $ABx = BAx$ and $B^*x \varepsilon \mathfrak{D}(A)$, $AB^*x = B^*Ax$. Thus $B \varepsilon \mathbf{A}'$, and $\mathbf{A}'$ is weakly closed. The relation $A = A^{**}$, holding here for every A in $\mathbf{A}$, implies that $(\mathbf{A}^{**})' = \mathbf{A}'$. However we also have $\mathbf{A}' \subset (\mathbf{A}^*)' \subset (\mathbf{A}^{**})' \subset \mathbf{A}'$ so that $\mathbf{A}' = (\mathbf{A}^*)'$, as we wished to show.

Starting with $\mathbf{A}$, we can form the successive commutants $\mathbf{A}', \mathbf{A}'', \mathbf{A}''', \mathbf{A}'''', \ldots$. We find that only the first three terms of this sequence can be distinct. More precisely, we can prove the following theorem.

THEOREM 14. *For any set* $\mathbf{A}$ *we have* $\mathbf{A}' \subset \mathbf{A}'''$, $\mathbf{A}'' = \mathbf{A}''''$. *The inclusion* $\mathbf{A} \subset \mathbf{A}''$ *is valid if and only if* $\mathbf{A}$ *consists of bounded linear operators. The identity* $\mathbf{A}' = \mathbf{A}'''$ *is valid if and only if there exists a set* $\mathbf{B}$ *of bounded linear operators such that* $\mathbf{A}' = \mathbf{B}'$; *in particular, it is valid whenever* $\mathbf{A}$ *consists of closed linear operators.*

PROOF. It is evident that for any two sets **A** and **B** the inclusion $\mathbf{A} \subset \mathbf{B}$ implies $\mathbf{A}' \supset \mathbf{B}'$. If **A** consists of bounded linear operators, it is clear from the definition of $\mathbf{A}'$ and the relation $(\mathbf{A}^*)' = \mathbf{A}'$ that $\mathbf{A} \subset \mathbf{A}''$, i.e. that every operator in **A**, together with its adjoint, commutes with every operator in $\mathbf{A}'$ and hence belongs to $\mathbf{A}''$. Applying this result to $\mathbf{A}'$, where **A** is arbitrary, we conclude that $\mathbf{A}' \subset \mathbf{A}'''$. Again restricting **A** to consist of bounded linear operators, we infer from the inclusion $\mathbf{A} \subset \mathbf{A}''$ that $\mathbf{A}' \supset \mathbf{A}'''$. In this case, therefore, we have $\mathbf{A}' = \mathbf{A}'''$. Reverting once more to the general case and applying the last result to $\mathbf{A}'$, where **A** is arbitrary, we see that $\mathbf{A}'' = \mathbf{A}''''$. It is then evident that in the sequence $\mathbf{A}', \mathbf{A}'', \mathbf{A}''', \ldots$ only $\mathbf{A}', \mathbf{A}'', \mathbf{A}'''$ can be distinct. When $\mathbf{A}' = \mathbf{A}'''$ we can write $\mathbf{A}' = \mathbf{B}'$ where $\mathbf{B} = \mathbf{A}''$ consists of bounded linear operators. On the other hand, if such a set **B** exists we have $\mathbf{B}' = \mathbf{B}'''$, as observed above, and thus conclude that $\mathbf{A}' = \mathbf{B}' = \mathbf{B}''' = \mathbf{A}'''$. The preceding theorem shows that such a set **B** exists when **A** consists of closed linear operators. We have already seen that, if **A** consists of bounded linear operators, then $\mathbf{A} \subset \mathbf{A}''$. Since $\mathbf{A}''$ consists of bounded linear operators by definition, this inclusion cannot be valid in any other case.

Geometrically it is evident that the various problems under consideration here cannot be advanced without the introduction of certain constructive procedures. One such procedure, which seems to be a particularly natural one, is to consider the closed linear subspace generated from a single vector by repeated applications of the operators A and A^*, $A \in \mathbf{A}$, and the operations of vector algebra. This procedure is evidently not of great interest in the most general cases, because the operators in **A** need not be capable of iteration. For this reason we shall confine our attention to the case where **A** consists of bounded linear operators. Under this assumption it is evident that the closed subspace generated from a given vector x in the manner indicated is the closure of the set consisting

of all vectors $P(I, A_1, \dots, A_n, A_1^*, \dots, A_n^*)x$, $A_1 \varepsilon \mathbf{A}, \dots$, $A_n \varepsilon \mathbf{A}$, $n = 0, 1, 2, \dots$, where P denotes a polynomial with coefficients in the scalar system for $\mathfrak{H}$ and without constant term. Let us denote this subspace as $\mathfrak{M}(x, \mathbf{A})$. We shall use the following terminology.

Definition 6. *The closed subspace $\mathfrak{M}(x, \mathbf{A})$, where $x \varepsilon \mathfrak{H}$ and $\mathbf{A}$ is a set of bounded linear operators is called a monogenic subspace*† *—specifically, the monogenic subspace generated from x by $\mathbf{A}$. The closed subspace $\mathfrak{H}$ is said to be monogenic with respect to $\mathbf{A}$ if there exists a vector x such that $\mathfrak{H} = \mathfrak{M}(x, \mathbf{A})$.*

It is now easy to prove the following results concerning monogenic subspaces.

Theorem 15. *If $\mathfrak{M}(x, \mathbf{A})$ is a monogenic subspace, then it contains the vector x, reduces every operator in $\mathbf{A}$, and has as its projection an operator in $\mathbf{A}'$. Among the closed subspaces which contain x and reduce every A in $\mathbf{A}$ the monogenic subspace $\mathfrak{M}(x, \mathbf{A})$ is the least. A set of mutually orthogonal non-trivial monogenic subspaces, all contained in a fixed closed subspace $\mathfrak{M}$ which reduces every A in $\mathbf{A}$ is maximal if and only if $\mathfrak{M}$ is their direct sum; and the given set is always part of at least one such maximal set.*

Proof. We evidently have $x = Ix \varepsilon \mathfrak{M}(x, \mathbf{A})$. If $A \varepsilon \mathbf{A}$ then A and A^* both map into itself the set of all vectors of the form $P(I, A_1, \dots, A_n, A_1^*, \dots, A_n^*)x$. Since this set is everywhere dense in $\mathfrak{M}(x, \mathbf{A})$, the continuity of the operators A and A^* implies that they also map $\mathfrak{M}(x, \mathbf{A})$ into itself. Thus $\mathfrak{M}(x, \mathbf{A})$ reduces A. Its projection, of course, then commutes with A and by definition therefore belongs to $\mathbf{A}'$. If a closed subspace $\mathfrak{M}$ contains x and reduces every A in $\mathbf{A}$, it is evident that $\mathfrak{M}$ contains every vector $P(I, A_1, \dots, A_n, A_1^*, \dots, A_n^*)x$, $A_1 \varepsilon \mathbf{A}, \dots, A_n \varepsilon \mathbf{A}$, and hence also the monogenic subspace $\mathfrak{M}(x, \mathbf{A})$. In any subspace $\mathfrak{M}$ which reduces every A in $\mathbf{A}$, consider a set of mutually orthogonal monogenic subspaces, each non-trivial in the sense that it contains some vector other

† We prefer this term to the term "cyclic subspaces" used by several authors.

than o. Unless $\mathfrak{M}$ contains a vector $x \neq 0$ which is orthogonal to every one of the given monogenic subspaces, their direct sum coincides with $\mathfrak{M}$. If there exists such a vector x, then $\mathfrak{M}(x, \mathbf{A}) \neq (0)$ is a monogenic subspace orthogonal to all the given ones. Indeed if $\mathfrak{N}$ is any one of the given subspaces, its orthogonal complement $\mathfrak{N}^*$ contains x and reduces every A in $\mathbf{A}$. Consequently we have $\mathfrak{M}(x, \mathbf{A}) \subset \mathfrak{N}^*$. Since $\mathfrak{M}(x, \mathbf{A}) \subset \mathfrak{M}$, the given set is not maximal in $\mathfrak{M}$. Finally, we note that whenever the given set is not maximal in $\mathfrak{M}$, the maximal principle (Zorn's lemma) shows that it is part of a maximal such set.

The theorem just established evidently provides a method for constructing all the closed subspaces $\mathfrak{M}$ which reduce every A in $\mathbf{A}$ and hence for determining all the projections in $\mathbf{A}'$. In the extreme case where $\mathfrak{M}(x, \mathbf{A}) = \mathfrak{H}$ for every $x \neq 0$, the only projections in $\mathbf{A}'$ are O and I; and in this case we are still far from having the means for finding out what operators belong to $\mathbf{A}'$. Thus Theorem 15 needs to be supplemented by other considerations before our knowledge of commutants can be regarded as adequate. One of the useful services of the spectral theory is to show in the complex case that under the present hypotheses the projections in $\mathbf{A}'$ are sufficiently numerous to generate it. In the example just cited, it can then be seen that $\mathbf{A}'$ consists entirely of operators of the form αI.

An immediate application of the concepts introduced in Definition 6 and Theorem 15 is based on the following lemma.

LEMMA 2. *If* $B \in \mathbf{A}''$, $x_1 \in \mathfrak{H}, \ldots, x_m \in \mathfrak{H}$, *and* $\varepsilon > 0$, *then there exists a polynomial operator* $C = P(I, A_1, \ldots, A_n, A_1^*, \ldots, A_n^*)$, $A_1 \in \mathbf{A}, \ldots, A_n \in \mathbf{A}$, *such that* $|Bx_k - Cx_k| < \varepsilon$ *for* $k = 1, \ldots, n$.

PROOF. When $m = 1$, the lemma merely asserts that $Bx_1 \in \mathfrak{M}(x_1, \mathbf{A})$. To verify the latter relation, we recall

that the projection of $\mathfrak{M}(x_1, \mathbf{A})$ is in $\mathbf{A}'$ and hence commutes with $B \in \mathbf{A}''$. Thus B is reduced by $\mathfrak{M}(x_1, \mathbf{A})$ and $Bx_1 \in \mathfrak{M}(x_1, \mathbf{A})$. The general case is brought back to this one by an artifice due to von Neumann [3]. We consider in the direct sum of m replicas of $\mathfrak{H}$ the set $\mathbf{A}_m$ of all operators $A_m : (x_1, \ldots, x_m) \to (Ax_1, \ldots, Ax_m)$ where $A \in \mathbf{A}$. Simple matrix calculations, similar to those used elsewhere in this paper, show that $\mathbf{A}_m$ consists of all the operators $B_m : (x_1, \ldots, x_m) \to (Bx_1, \ldots, Bx_m)$ where $B \in \mathbf{A}''$. Details may be found in von Neumann's paper [3]. The desired inequalities are then obtained.

With the help of this lemma we can characterize $\mathbf{A}''$ in terms of $\mathbf{A}$.

THEOREM 16. *Let $\mathbf{A}$ be a set of bounded linear operators. Let $\mathbf{P}$ be the set of all polynomial operators $P(I, A_1, \ldots, A_n, A_1^*, \ldots, A_n^*)$, $A_1 \in \mathbf{A}, \ldots, A_n \in \mathbf{A}$, with coefficients in the scalar system for $\mathfrak{H}$ and without constant term. Then $\mathbf{P}$ is part of the set of all bounded real-linear operators in $\mathfrak{H}$, and has with respect to the strong topology in the latter set a closure which will be denoted as $\mathbf{B}$. Let $\mathbf{S}$ be the set of all operators aI. Then $\mathbf{A}'' = \mathbf{B} \cap \mathbf{S}'$. In the real and complex cases, this relation reduces to the simpler one $\mathbf{A}'' = \mathbf{B}$.*

PROOF. The preceding lemma shows that $\mathbf{A}'' \subset \mathbf{B}$. Since $\mathbf{S}'$ is the set of all bounded linear operators, we also have $\mathbf{A}'' \subset \mathbf{S}'$ and hence $\mathbf{A}'' \subset \mathbf{B} \cap \mathbf{S}'$. If $B \in \mathbf{B}$ it is a bounded real-linear operator. If C is an arbitrary operator in $\mathbf{A}'$, then every operator in $\mathbf{P}$ commutes with both C and C^* since this is the case for every operator in $\mathbf{A}$ and for every operator in $\mathbf{S}$. In view of the fact that in the strong topology the product of two operators is separately continuous in its factors, we see that B also must commute with both C and C^*. If we know further that $B \in \mathbf{S}'$, then we can add that B is linear and that its adjoint B^* commutes with C since $B^*C = (C^*B)^* = (BC^*)^* = CB^*$. Thus $B \in \mathbf{B} \cap \mathbf{S}'$ implies $B \in \mathbf{A}''$. Accordingly we have $\mathbf{B} \cap \mathbf{S}' = \mathbf{A}''$, as we wished to prove. In the real and

complex cases the operators in **P** are linear rather than merely real-linear, in other words, $\mathbf{P} \subset \mathbf{S}'$. Since the strong topology in $\mathbf{S}'$ coincides with the topology induced in $\mathbf{S}'$ as a part of the space of all bounded *real-linear* operators under the strong topology, we see that in these cases $\mathbf{B} \subset \mathbf{S}'$ and $\mathbf{A}'' = \mathbf{B}$.

THEOREM 17. *Let* **A** *be a set of bounded linear operators, let* $\mathfrak{M}$ *be the set of all vectors* x *such that* $Ax = A^*x = 0$ *for every* A *in* **A**, *and let* E *be the projection of* $\mathfrak{H}$ *on the orthogonal complement of* $\mathfrak{M}$. *Then* $E \in \mathbf{A}' \cap \mathbf{A}''$ *and* $EA = AE = A$ *for every* A *in* **A**. *If* **C** *is the strong closure of the set of all polynomial operators* $Q(A_1, \ldots, A_n, A_1^*, \ldots A_n^*)$, $A_1 \in \mathbf{A}, \ldots, A_n \in \mathbf{A}''$ *where* Q *is a polynomial with coefficients in the scalar system for* $\mathfrak{H}$ *and without constant term, then* $B \in \mathbf{C} \cap \mathbf{S}'$ *if and only if* $EB = BE = B \in A''$.

PROOF. Since A and A^* are continuous linear operators, $\mathfrak{M}$ is a closed subspace. Its projection is therefore $I-E$. Every A in **A** is reduced by $\mathfrak{M}$ since A and A^* both map $\mathfrak{M}$ onto (o). Thus $I-E$ and E are in $\mathbf{A}'$. If $x \in \mathfrak{M}$ and $C \in \mathbf{A}'$, then $ACx = CAx = 0$ and $A^*Cx = CA^*x = 0$, so that $Cx \in \mathfrak{M}$. Thus if $C \in \mathbf{A}'$, we have $C^* \in \mathbf{A}$; and both C and C^* map $\mathfrak{M}$ into itself. If follows that $\mathfrak{M}$ reduces every C in $\mathbf{A}'$. Accordingly $I-E$ and E are in $\mathbf{A}''$. Since $\mathfrak{M}$ is the range of $I-E$ we evidently have $(I-E)A = A(I-E) = 0$ or $EA = AE = A$, for any A in **A**. By taking the adjoints of the operators in the last equality, we find that $EA^* = A^*E = A^*$ as well. Now let $B \in \mathbf{C} \cap \mathbf{S}'$. Since $\mathbf{C} \subset \mathbf{B}$, the preceding theorem shows $B \in \mathbf{A}''$. Since $EC = CE = C$ holds for any polynomial operator $C = Q(A_1, \ldots, A_n, A_1^*, \ldots, A_n^*)$ of the type under consideration, the relation $B \in \mathbf{C}$ implies through a continuity argument that $EB = BE = B$. On the other hand, suppose that $EB = BE = B \in \mathbf{A}''$. We consider first the case where $E \neq I$. Let $x_1, \ldots, x_m$ and $\varepsilon > 0$ be given. Then we choose $x_0 \in \mathfrak{M}$ so that $|x_0| = 1$, and $\eta > 0$ so that $\eta(1+|x_k|) \leq \varepsilon$ for $k = 1, \ldots, m$. Since $B \in \mathbf{A}''$, Lemma 2 asserts the existence

of a pylynomial operator $D = \alpha I + C$ where $C = Q(A_1 \ldots, A_n, A_1^*, \ldots, A_n^*)$ as above, such that $|Bx_k - Dx_k| < \eta$ for $k = 0, 1, \ldots, m$. When $k = 0$, the corresponding inequality reduces to $|\alpha| < \eta$ since $Ex_0 = 0$, $Bx_0 = BEx_0 = 0$, $Cx_0 = CEx_0 = 0$; and $|x_0| = 1$. Hence for $k = 1, \ldots, m$ we have $|Bx_k - Cx_k| < \eta + |\alpha|\,|x_k| \leq \varepsilon$. This means that $B \varepsilon \mathbf{C}$. As B is bounded linear, it is also in $\mathbf{S}'$. Thus $B \varepsilon \mathbf{C} \cap \mathbf{S}'$ as we wished to show. Finally, we treat the case where $E = I$. We consider the set A_2 of all operators in $\mathfrak{H} \oplus \mathfrak{H}$ with matrices of the form $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, $A \varepsilon \mathbf{A}$. The associated projection E_2 is evidently given by the matrix $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. A matrix commutes with that for E_2 if and only if it is a diagonal matrix. Hence the operators in $\mathbf{A}_2'$ and $\mathbf{A}_2''$ all have diagonal matrices. It is then easy to see that a diagonal matrix is in $\mathbf{A}_2'$ if and only if its upper left component is in $\mathbf{A}'$. The operator B_2 with matrix $\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ is evidently in $\mathbf{A}_2''$ and satisfies the identity $E_2 B_2 = B_2 E_2 = B_2$. Consequently $B_2 \varepsilon \mathbf{C}_2$ by the results already proved. This obviously means that $B \varepsilon \mathbf{C}$. Since B is a bounded linear operator, we also have $B \varepsilon \mathbf{C} \cap \mathbf{S}'$, as we wished to prove.

We may note the following well-known specialization of Theorem 17.

Corollary. *In the real or the complex case, let* **A** *be a set of bounded linear operators which is a self-adjoint linear algebra over the scalar system for* $\mathfrak{H}$. *Then* **A** *is weakly closed if and only if it is strongly closed.*

Proof. The strong closure of **A** is now identical with **C**. Hence **A** is strongly closed if and only if **A** = **C**. On the other hand, since the products $EB = BE$ are weakly continuous in B the identities $EB = BE = B$ specify **C** as a weakly closed part of the weakly closed set

A$''$. Thus if **A** is strongly closed it is also weakly closed. The converse is trivial since the strong topology is finer than the weak.

4. **Abelian sets of operators.** The final section of this paper will be devoted to a discussion of commutativity for non-bounded operators. It is very difficult to decide under just what conditions two non-bounded operators should be considered to commute. It may be that in practice no completely satisfactory or final decision will be reached except under special circumstances. In any case, it is obvious that no workable definition can be formulated which does not lay certain explicit requirements on the domains of the operators. The surprising fact is that these requirements do not always need to be quite so heavy as might be expected *a priori*. The success of the definition of normality which we gave above is an illustration.

The motive behind most definitions of commutativity is that of erecting criteria which will depend mainly on commutativity properties of bounded linear operators and as little as possible on qualifications concerning the domains of the operators being tested. It should also be stated that most definitions are concerned not with simple commutativity but rather with the simultaneous commutativity of operators and their adjoints. This will be the case in the ensuing discussion. If **A** is a set of bounded linear operators the relations noted in Theorem 14 show that the following propositions are equivalent: $\mathbf{A} \subset \mathbf{A}'$; $\mathbf{A}''$ is a commutative algebra; the operators in **A** and their adjoints commute with one another. It is therefore natural to consider a set **A** of quite arbitrary operators to be in some sense commutative if $\mathbf{A}''$ is commutative, that is, equivalently, if $\mathbf{A}'' \subset \mathbf{A}'''$. At the same time, we should not expect this condition to be workable without the support of some further condition which will guarantee that the operators in **A** have fairly

large domains. Conditions phrased in terms of the *individual* domains would not have any very great likelihood of overcoming the difficulties which take their origin in the *mutual* relations of the operators under scrutiny. Hence it appears that the supplementary condition we shall introduce here—namely that $\mathbf{A}' = \mathbf{A}'''$ (instead of merely $\mathbf{A}' \subset \mathbf{A}'''$)—is a reasonable one. It should be noted, of course, that in view of the definition of commutativity any requirement tending to enlarge $\mathbf{A}'$ tends to enlarge the domains of the operators left in $\mathbf{A}$. Our fundamental definition therefore reads as follows.

Definition 7. *A set* $\mathbf{A}$ *of operators in* $\mathfrak{H}$ *is said to be abelian if and only if* $\mathbf{A}' \supset \mathbf{A}''' \supset \mathbf{A}''$—*or, equivalently, if and only if* $\mathbf{A}''$ *is a commutative algebra and* $\mathbf{A}' = \mathbf{A}'''$. *An operator* A *is said to be abelian if and only if the set* (A) *is abelian.*

In view of the preceding discussion, and especially of Theorems 13 and 14, we may note the following facts.

THEOREM 18. *When* $\mathbf{A}$ *consists of closed linear operators, it is abelian if and only if* $\mathbf{A}''$ *is commutative; and when* $\mathbf{A}$ *consists of bounded linear operators, it is abelian if and only if the operators in* $\mathbf{A}$, *together with their adjoints, commute with one another.*

We shall now establish a few general properties of abelian sets.

THEOREM 19. *Let* $\mathbf{B}$ *consist of closed linear operators and let* $\mathbf{A}''$ *be an abelian set. Then if* $\mathbf{B} \subset \mathbf{A}$ *or* $\mathbf{B}' \supset \mathbf{A}'$, $\mathbf{B}$ *is an abelian set. If* $\{\mathbf{A}_\alpha\}$ *is a family of abelian sets such that the operators in the various commutative algebras* $\mathbf{A}_\alpha''$ *are mutually commutative, then the union* $\mathbf{A} = \bigcup_\alpha \mathbf{A}_\alpha$ *is an abelian set.*

PROOF. By hypothesis, $\mathbf{B}' = \mathbf{B}'''$. Thus $\mathbf{B} \subset \mathbf{A}$ implies $\mathbf{B}' \supset \mathbf{A}'$, which in turn implies $\mathbf{B}'' \subset \mathbf{A}''$. Since $\mathbf{A}''$ is commutative, so is $\mathbf{B}''$. When $\mathbf{A} = \bigcup_\alpha \mathbf{A}_\alpha$, we have $(\bigcup_\alpha \mathbf{A}_\alpha'')' = \bigcap_\alpha \mathbf{A}_\alpha''' = \bigcap_\alpha \mathbf{A}_\alpha' = \mathbf{A}'$, so that $\mathbf{A}' = \mathbf{A}'''$ in accordance with Theorem 14 and $\mathbf{A}'' = (\bigcup_\alpha \mathbf{A}_\alpha'')''$ is commutative by hypothesis and in accordance with the preceding theorem.

quaternionic case. The difficulty with the quaternionic case is encountered in the first lemma below, but does not appear in later parts of the discussion. This lemma can render useful service in other connections, so that a modification of it would apparently be needed in mastering the quaternionic cases. The second lemma is an easy consequence of the first and leads directly to our demonstration of the S-O-M Theorem. The third is essential in this last step.

LEMMA 3. *Let* $\mathfrak{H}$ *be a real or complex Hilbert space monogenic with respect to a vector* x *and an abelian set* $\mathbf{A}$ *of bounded linear operators, so that* $\mathfrak{H} = \mathfrak{M}(x, \mathbf{A})$. *For any fixed vector* y, *let* S *be the operator defined by the correspondence* $Dx \rightarrow Dy$, *where* $D = P(I, A_1, \ldots, A_n, A_1^* \ldots, A_n^*)$, $A_1 \in \mathbf{A}$, $\ldots, A_n \in \mathbf{A}$, *is a polynomial operator with coefficients in the scalar system for* $\mathfrak{H}$. *Then* $\mathbf{A}' = \mathbf{A}''$; *and* S^{**} *exists and is a normal operator for which the representation* $S^{**} = B^{-1}C$ *of Theorem* 9 *is such that* $B \in \mathbf{A}''$ *and* $C \in \mathbf{A}''$. *In particular, any vector* y *in* $\mathfrak{H}$ *can be connected with* x *by a relation* $By = Cx$, *where* B *and* C *are operators in* $\mathbf{A}''$ *enjoying the properties enumerated in Theorem* 9.

PROOF. By definition, the present hypotheses mean that there exists a sequence $\{Y_n\}$ of polynomial operators such that $\{Y_n x\}$ converges to y. Since $Y_n \in \mathbf{A}''$ it is evident that Y_n is a normal operator. Consequently, $|Y_m x - Y_n x| = |(Y_m - Y_n)x| = |(Y_m - Y_n)^* x| = |Y_m^* - Y_n^* x|$. The sequence $\{Y_n x\}$ therefore converges to a vector z. We consider simultaneously the operators $S = S_y$: $Dx \rightarrow Dy$ and S_z: $Dx \rightarrow Dz$. We have $(S_y D_1 x)(D_2 x) = (D_1 y)(D_2 x) = \lim_{n\to\infty} (D_1 Y_n x)(D_2 x) = \lim_{n\to\infty} (D_1 x)(D_2 Y_n^* x) = (D_1 x)(D_2 z) = (D_1 x)(S_z D_2 x)$ for any D_1 and D_2. Since the vectors Dx consitute an everywhere dense set, we see that $D_1 x = 0$ implies $D_1 y = D_1 z = 0$ and that S_y and S_z are linear operators with adjoints $S_y^* \supset S_z$, $S_z^* \supset S_y$. Moreover, S_y^* and S_z^* are regular operators since $\mathfrak{D}(S_y)$

and $\mathfrak{D}(S_z)$ are everywhere dense sets. It is not difficult to ascertain some of the commutation properties of these operators. First we shall show that if $R \in \mathbf{A}'$ then $S_z R D_1 x = R S_y D_1 x$ for any D. For any D_1 and D_2, we have

$$\begin{aligned}(S_z{}^* R D_1 x)\, D_2 x &= (R D_1 x)\ (S_z D_2 x) = (D_1 x)\ (R^* D_2 z) \\ &= \lim_{n\to\infty} (D_1 x)\,(R^* D_2\, \Upsilon_n{}^* x) = \lim_{n\to\infty} (D_1\, \Upsilon_n\, x)\ (R^* D_2 x) \\ &= (D_1\, y)\ (R^* D_2 x) = (S_y D_1 x)\ (R^* D_2 x) \\ &= (R S_y D_1 x)\, D_2 x,\end{aligned}$$

and hence the result at which we aimed. Next we can use this information to show that $\mathbf{A}' = \mathbf{A}''$. Let R and T be arbitrary operators in $\mathbf{A}'$. If we take $y = Tx$ it is easy to see that $T \supset S_y$ since $TDx = DTx = Dy$. Accordingly we have $S_y{}^* \supset T^*$, $S_y{}^{**} = T$, $S_z \subset S_y{}^* = T^*$, $S_z{}^* \supset T$. Hence the relation derived above reduces for $y = Tx$ to $TRD_1 x = RTD_1 x$. Thus the continuous operators TR and RT, being equal on an everywhere dense part of $\mathfrak{H}$, must coincide.

As the same statement may be made about the operators $R \in \mathbf{A}'$ and $T^* \in \mathbf{A}'$ we see that $T \in \mathbf{A}'$ implies $T \in \mathbf{A}''$. Thus we have $\mathbf{A}' \subset \mathbf{A}''$, whence $\mathbf{A}'' \supset \mathbf{A}''' = \mathbf{A}'$ and $\mathbf{A}' = \mathbf{A}''$. Next we shall show that any operator R in $\mathbf{A}'$ commutes with S^{**}. Since $\mathbf{A}' = \mathbf{A}''$, Lemma 2 is applicable and provides a sequence $\{D_n\}$ of polynomial operators such that $\{D_n D_1 x\}$ and $\{D_n S_y D_1 x\}$ converge to $RD_1 x$ and $RS_y D_1 x$ respectively. Since $D_n \in \mathbf{A}'$ and $D_n D_1 x \in \mathfrak{D}(S_y)$ we see that $S_y D_n D_1 x = S_z{}^* D_n D_1 x = D_n S_y D_1 x$. The fact that $S_y{}^{**} \supset S_y$ now leads to the conclusion that $RD_1 x \in \mathfrak{D}(S_y{}^{**})$ and $S_y{}^{**} RD_1 x = \lim_{n\to\infty} S_y D_n D_1 x = \lim_{n\to\infty} D_n S_y D_1 x = RS_y D_1 x$ for every D_1. If w is an arbitrary vector in the domain of S^{**}, we can find a sequence $\{D_n\}$ such that $\{D_n x\}$ and $\{S_y D_n x\} = \{D_n y\}$ converge to w and $S^{**} w$ respectively: for the graph of S^{**} is the closure of the graph of the linear operator S_y. Then we see that $\{RD_n x\}$ and $\{S^{**} RD_n x\} = \{RSD_n\}$ converge

to Rw and $RS^{**}w$ respectively. Hence $Rw \in \mathfrak{D}(S^{**})$ and $S^{**}Rw = RS^{**}w$. Thus $S^{**} = S_y^{**}$ is a regular operator such that $(S^{**})' \supset \mathbf{A}'$, where $\mathbf{A}$ is an abelian family. Accordingly S_y^{**} is normal, as we have seen in Theorem 20; and S_y^{*} is also a normal operator, with domain equal to that of S_y^{**}. The fact that $S_y^{**} \subset S_z^{*}$, $S_z^{**} \subset S_y^{*}$, $\mathfrak{D}(S_y^{*}) = \mathfrak{D}(S_y^{**})$, $\mathfrak{D}(S_z^{*}) = \mathfrak{D}(S_z^{**})$ clearly implies the relations $S_y^{**} = S_z^{*}$, $S_z^{**} = S_y^{*}$. If we now apply Theorem 9 to the normal operator S^{**} we immediately see that the operators B and C, being components of the characteristic matrix for S^{**}, are in $(S^{**})'' \subset \mathbf{A}''$. The fact that $y = S_y x = B^{-1}Cx$ implies $y \in \mathfrak{D}(B)$ and $By = Cx$, as we wished to show.

Lemma 4. *Let x and $\mathbf{A}$ be as in Lemma 3. Let T be an arbitrary operator such that $x \in \mathfrak{D}(T)$ and $(T)' \supset \mathbf{A}'$. Then S_y^{**}, where $y = Tx$, is an extension of T.*

Proof. It suffices to show that, if $B = B_y$ and $C = C_y$ as in Lemma 3, then $B_y Tw = C_y w$ for every $w \in \mathfrak{D}(T)$: for $S_y^{**} = B_y^{-1} C_y$. For a given such vector w, let S_w, B_w, and C_w be determined as in Lemma 3, so that $B_w w = C_w x$. Since B_y, C_y, B_w and C_w are in $\mathbf{A}'' = \mathbf{A}'$, we have

$$B_w B_y Tw = B_y TB_w w = B_y TC_w x = C_w B_y Tx$$
$$= C_w C_y x = C_y C_w x = C_y B_w x = B_w C_y w.$$

As B_w is invertible, $B_y Tw = C_y w$; and the proof is complete.

Lemma 5. *In a real or a complex Hilbert space $\mathfrak{H}$, let $\mathbf{C}$ be any abelian set of bounded linear operators; let $\mathbf{A}$ be any set of operators such that $\mathbf{A}' \supset \mathbf{C}'$; and let x be an arbitrary vector. The closed subspace $\mathfrak{M} = \mathfrak{M}(x, \mathbf{C})$ reduces every operator in $\mathbf{C}$ or in $\mathbf{A}$. If $\mathbf{C}_0$ and $\mathbf{A}_0$ are the sets of operators in $\mathfrak{M}$ obtained by restricting to $\mathfrak{M}$ the operators in $\mathbf{C}$ and in $\mathbf{A}$ respectively, then $\mathbf{A}_0' \supset \mathbf{C}_0' \supset \mathbf{C}_0''$, $\mathbf{C}_0$ is an abeliam set, and $\mathfrak{M}$ is monogenic with respect to $\mathbf{C}_0$.*

Proof. By Theorem 15 the projection E of $\mathfrak{H}$ on $\mathfrak{M}$ is an operator in $\mathbf{C}'$ and hence also in $\mathbf{A}'$. Consequently $\mathfrak{M}$ reduces every operator in $\mathbf{C}$ or in $\mathbf{A}$. Let us observe

that if two operators reduced by $\mathfrak{M}$ are adjoint in $\mathfrak{H}$ then their restrictions to $\mathfrak{M}$ are adjoint in $\mathfrak{M}$. It is then readily seen that $\mathbf{C}_0$ is an abelian set with $\mathfrak{M}(x, \mathbf{C}_0) = \mathfrak{M}(x, \mathbf{C}) = \mathfrak{M}$. Lemma 3 then shows that $\mathbf{C}_0' = \mathbf{C}_0'' \supset \mathbf{C}_0$. If B_0 is an arbitrary operator in $\mathbf{C}_0'$ and B_0^* its adjoint in $\mathfrak{M}$, these operators have the respective bounded linear extensions $B_0 E$ and $B_0^* E$ in $\mathfrak{H}$. The latter operators, being adjoint to each other, may be denoted by B and B^* respectively. If C is and operator in $\mathbf{C}$ and C_0 its restriction to $\mathfrak{M}$, we have $C_0 \epsilon \mathbf{C}_0$ and hence $CB = CB_0 E = C_0 B_0 E = B_0 C_0 E = BCE = BEC = BC$. Similarly $CB^* = B^*C$. Thus $B \epsilon \mathbf{C}' \subset \mathbf{A}'$. If $A \epsilon \mathbf{A}$ and A_0 is the restriction of A to $\mathfrak{M}$, we therefore have for any x in $\mathfrak{D}(A_0)$ the relations $B_0 x = Bx \epsilon \mathfrak{D}(A) \cap \mathfrak{M} = \mathfrak{D}(A_0)$ and $A_0 B_0 x = ABx = BAx = B_0 A_0 x$. Thus B_0 commutes with A_0. Similarly B_0^* commutes with A_0. It follows that $B_0 \epsilon \mathbf{A}_0'$ and hence that $\mathbf{C}_0' \subset \mathbf{A}_0'$, as we wished to prove.

We are now ready to establish the S-O-M Theorem. In reality this theorem consists essentially in removing some of the restrictive conditions in Lemma 4.

THEOREM 21. (S-O-M Theorem). *Let $\mathfrak{H}$ be a real or a complex Hilbert space; let $\mathbf{C}$ be any abelian set; and let $\mathbf{A}$ be any set of operators such that $\mathbf{A}' \supset \mathbf{C}'$. Then each operator A in $\mathbf{A}$ has in the closed subspace generated by $\mathfrak{D}(A)$ a unique normal extension $\bar{A}$. If $\mathfrak{D}(A)$ generates $\mathfrak{H}$, then $\bar{A}$ is represented in the form $B^{-1}C$ of Theorem 9 with $B \epsilon \mathbf{C}''$ and $C \epsilon \mathbf{C}''$. The set $\bar{\mathbf{A}}$ of all the extensions $\bar{A}, A \epsilon \mathbf{A}$, is an abelian set. Whenever $\mathbf{A}$ is an abelian set the above hypotheses are satisfied, with $\mathbf{C} = \mathbf{A}$.*

PROOF. Since the relation $\mathbf{C}' = \mathbf{C}'''$ permits us to replace $\mathbf{C}$ by $\mathbf{C}''$, there is no loss of generality in supposing as we shall do below that $\mathbf{C}$ consists of bounded linear operators. Our chief effort will be to show that for any A in $\mathbf{A}$ the characteristic matrix $< P_{ik}(A) >$ satisfies condition (4). Once this has been done all the statements of the theorem are either obvious, like the last one, or easily

derived. Denoting by $\mathbf{B}$ the set of all the components $P_{ik}(A)$, we see that $\mathbf{B}' \supset \mathbf{A}' \supset \mathbf{C}'$ and hence that $\mathbf{B} \subset \mathbf{B}'' \subset \mathbf{C}'' \subset \mathbf{C}' \subset \mathbf{A}' \subset \mathbf{B}'$. Thus the components $P_{ik}(A)$ are normal and commute with one another. If condition (4) has been verified we know from Theorem 8 that there exists in the subspace generated by $\mathfrak{P}(A)$ a closed linear operator $\bar{A}$ which as an operator in that subspace is normal and as an operator in $\mathfrak{H}$ has $< P_{ik}(A) >$ as its characteristic matrix. $\bar{A}$ is obviously an extension of A. Theorem 20 shows immediately that $\bar{\mathbf{A}}$ is an abelian set. In the special case that $\mathfrak{P}(A)$ generates $\mathfrak{H}$, we recall that in the representation $\bar{A} = B^{-1}C$ of Theorem 9 the operators B and C are among the components P_{ik} $(\bar{A})$ and hence are now in $\mathbf{B}$ and in $\mathbf{C}''$. It is also easy to verify that the extension $\bar{A}$ is unique. In the c'osed subspace generated by $\mathfrak{P}(A)$ the argument above shows that $\bar{A} = A^{**}$, in accordance with Theorem 8. Hence if S is any normal extension of A in that subspace, we have $S \supset A^{**}$ and $S^* \subset A^*$. The relations $\mathfrak{P}(S) = \mathfrak{P}(S^*)$ and $\mathfrak{P}(A^{**}) = \mathfrak{P}(A^*)$ then show that $S = A^{**} = \bar{A}$. We turn, then, to the discuss'on of property (4). Considering the matrix $< P_{ik} > = < P_{ik}(A) >$ for a fixed operator A in $\mathbf{A}$, we form the set $\mathfrak{M}$ of all vectors x such that $P_{12}x = (I - P_{22})\, x = 0$. Since P_{12} is a normal operator with $P_{12}{}^* = P_{21}$, we see by Theorem 7 that $P_{12}\, x = 0$ if and only if $P_{21}\, x = 0$. Now according to Theorem 17 the set $\mathfrak{N}$ is a closed subspace and the projection of $\mathfrak{H}$ upon it is an operator in $\mathbf{B}'' \subset \mathbf{C}'$. Thus $\mathfrak{N}$ reduces every operator in $\mathbf{C}$ and hence contains $\mathfrak{M} = \mathfrak{M}\ (x, \mathbf{C})$, $x \varepsilon \mathfrak{N}$, in accordance with Theorem 15. The projection of $\mathfrak{H}$ on $\mathfrak{M}$ is an operator in $\mathbf{C}' \subset \mathbf{A}' \subset \mathbf{B}'$. Hence $\mathfrak{M}$ reduces every operator in $\mathbf{A}$ or in $\mathbf{B}$. Applying the notation of Lemma 5 and denoting by A_0 the restriction of A to $\mathfrak{M}$, we see that $(A_0)' \supset \mathbf{A}_0{}' \supset \mathbf{C}_0{}'$. Thus Lemma 4 shows that A_0 has a closed linear extension in $\mathfrak{M}$. The characteristic matrix of A_0 in $\mathfrak{M}$ therefore has property (4). This matrix, since $\mathfrak{M}$ reduces every

operator in **A** or in **B**, is obtained by restricting the matrix $< P_{ik} >$ to $\mathfrak{M}$; and since $\mathfrak{M} \subset \mathfrak{N}$ it obviously has the form $\begin{pmatrix} P_{11} & 0 \\ 0 & I \end{pmatrix}$ in $\mathfrak{M}$. Since the premise of condition (4) is automatically satisfied for this matrix, we conclude that the only vector in $\mathfrak{M}$ is o. Thus $x \varepsilon \mathfrak{N}$ implies $x = 0$, in other words, condition (4) is verified by the original matrix $< P_{ik} >$. This completes the proof.

We may note a few useful corollaries.

COROLLARY 1. *If S and T are normal operators such that $S \subset T$, then $S = T$.*

PROOF. This special case of the theorem was established in the course of the demonstration above. The proof given there holds in the quaternionic case as well.

COROLLARY 2. *Under the hypotheses of Theorem 21 every operator A in* **A** *has a normal extension with domain which generates $\mathfrak{H}$.*

PROOF. If E is the projection of $\mathfrak{H}$ on the closed subspace generated by $\mathfrak{D}(A)$, then $\bar{A}E = E\bar{A}E$ is such an extension.

Finally we note the original version of the S-O-M Theorem.

COROLLARY 3. *Let $\mathfrak{H}$ be a complex separable Hilbert space; let H be a self-adjoint operator; and let A be an operator such that $(A)' \supset (H)'$. Then A has a normal extension which is, in the sense of the calculus of operators, a function of H. When $\mathfrak{D}(A)$ generates $\mathfrak{H}$ this extension is unique.*

PROOF. If $\mathfrak{D}(A)$ is empty, then O is an extension of A defined everywhere in $\mathfrak{H}$ and satisfying the relation $(O)' \supset (H)'$. If $\mathfrak{D}(A)$ is not empty, then $0 \varepsilon \mathfrak{D}(A)$ and $A0 = 0$: for $x \varepsilon \mathfrak{D}(A)$ and $O \varepsilon (H)' \subset (A)'$ imply $0 = Ox \varepsilon \mathfrak{D}(A)$ and $A0 = AOx = OAx = 0$. The operator AE, where E is the projection of $\mathfrak{H}$ on the closed subspace generated by $\mathfrak{D}(A)$, is an extension of A with domain which generates $\mathfrak{H}$. Now $E \varepsilon (A)'' \subset (H)'' \subset (H)' \subset (A)'$ since every operator in $(A)'$ carries $\mathfrak{D}(A)$ into itself and

therefore acts similarly on the closed subspace generated by $\mathfrak{D}(A)$. If C is any operator in $(A)'$ it commutes with A, E and hence AE. Thus $(AE)' \supset (A)' \supset (H)'$. Thus there in no loss of generality in supposing that $\mathfrak{D}(A)$ generates $\mathfrak{H}$. In this case $\bar{A} = B^{-1}C$ where $B \varepsilon (H)''$ and $C \varepsilon (H)''$ is the desired extension. Here B, C, and hence $\bar{A}$, are functions of H, in the indicated sense.

In conclusion we would like to point out that a weaker requirement than that of Definition 7 would consist in asking that $\mathbf{A}'' \subset \mathbf{A}'$. This condition would not, however, be sufficient for some of the more interesting results above. For example in Theorem 19 and in the application of Theorem 21 to abelian sets we have made essential use of the condition $\mathbf{A}' = \mathbf{A}'''$. When $\mathbf{A}$ consists of linear operators, $\mathbf{A}'$ is a linear algebra containing I; and the latter condition is equivalent to the condition that $\mathbf{A}'$ be closed in either the strong or the weak topology. Only in special circumstances, it would seem, can some such condition be eliminated. In the case of a single operator A, it is possible to follow von Neumann [3] in considering the case where $\mathfrak{D}(A)$ generates $\mathfrak{H}$, A^* is a regular operator, and $(A)''$ is commutative. In this case A^* and A^{**} are normal, as we have seen above; and any normal extension of A must be one of A^{**} and hence must coincide with the latter.

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